ISRAEL JOURNAL OF MATHEMATICS 153 (2006), 93–127

ACCIDENTAL PARABOLICS AND RELATIVELY HYPERBOLIC GROUPS

BY

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ABSTRACT

By constructing, in the relative case, objects analogous to Rips and Sela's canonical representatives, we prove that the set of conjugacy classes of images by morphisms without accidental parabolic, of a finitely presented group in a relatively hyperbolic group, is finite.

An important result of W. Thurston is:

THEOREM 0.1 ([T] 8.8.6): *Let S be any hyperbolic surface of finite area, and N any geometrically finite hyperbolic 3-manifold. There are only finitely many conjugacy classes of subgroups $G \subset \pi_1(N)$ isomorphic to $\pi_1(S)$ by an isomorphism which preserves parabolicity (in both directions).*

It is attractive to try to formulate a group-theoretic analogue of this statement: the problem is to find conditions such that the set of images of a group G in a group Γ is finite up to conjugacy.

If Γ is word-hyperbolic and G finitely presented, this has been the object of works by M. Gromov ([G] Theorem 5.3.C') and by T. Delzant [Del], who proves the finiteness (up to conjugacy) of the set of images by morphisms not factorizing through an amalgamation or an HNN extension over a finite group.

As a matter of fact, if a group G splits as $A *_C B$ and maps to a group Γ such that the image of C in Γ has a large centralizer, then in general, there are infinitely many conjugacy classes of images of G in Γ. Technically speaking, if h is the considered map, one can conjugate $h(A)$ by elements in the centralizer

Received May 27, 2003 and in revised form October 13, 2003

of $h(C)$, without modifying $h(B)$, hence producing new conjugacy classes of images. A similar phenomenon happens with HNN extensions.

We are interested here in the images of a group in a relatively hyperbolic group (for example, a geometrically finite Kleinian group). Our result, Theorem 0.2, gives a condition similar to the one of Thurston, ruling out the bad situation depicted above, and ensuring the expected finiteness.

Relatively hyperbolic groups were introduced by M. Gromov in [G], and studied by B. Farb [F] and B. Bowditch [B2], who gave different, but equivalent, definitions (see Definition 1.2 below, taken from [B2]). In Farb's terminology, we are interested in "relatively hyperbolic groups with the property BCP". The main example is the class of fundamental groups of geometrically finite manifolds (or orbifolds) with pinched negative curvature (see [B1], see also [F] for the case of finite volume manifolds). Sela's limit groups are hyperbolic relative to their maximal abelian non-cyclic subgroups, as shown in [D].

Definition: We say that a morphism from a group in a relatively hyperbolic group $h\colon G \to \Gamma$ has an accidental parabolic *either* if $h(G)$ is finite or parabolic in Γ, *or* if h can be factorized through a non-trivial amalgamated free product

$$\begin{array}{ccc} G & \xrightarrow{h} & \Gamma \\ & {\scriptstyle f}\searrow & \uparrow \\ & & A *_C B \end{array} \quad \text{or HNN extension} \quad \begin{array}{ccc} G & \xrightarrow{h} & \Gamma \\ & {\scriptstyle f}\searrow & \uparrow \\ & & A *_C \end{array} \quad \text{where } f \text{ is surjective, and}$$

the image of C is either finite or parabolic in Γ.

We prove the theorem:

THEOREM 0.2: *Let G be a finitely presented group, and Γ a relatively hyperbolic group. There are finitely many subgroups of Γ, up to conjugacy, that are images of G in Γ by a morphism without accidental parabolic.*

It would have been tempting to apply this to the mapping class group $Mod(S)$ of a surface, which is known to be "relatively hyperbolic", after the study of H. Masur and Y. Minsky of the complex of curves [MM]. If B is the base of a S-bundle, the study of homomorphisms $\pi_1(B) \to Mod(S)$ is important because it is directly related to the geometric Shafarevich conjecture (see the survey of C. McMullen [McM]). Unfortunately, the relative hyperbolicity of the mapping class group is to be understood in a weak sense: the property BCP, or equivalently the *fineness* (see Definition 1.2) is not fulfilled.

Also, note that Theorem 0.2 generalizes Theorem 0.1 in the case of closed surfaces: if a surface group $\pi_1(S)$ acts on a tree, at least one element associated to a simple curve in S fixes an edge. Therefore, if a morphism from $\pi_1(S)$ to

$\pi_1(N)$ (with notations of Theorem 0.1) has an accidental parabolic, it sends a simple curve of the closed surface S in a parabolic subgroup of $\pi_1(N)$.

We will begin by introducing the definitions and the objects of the theory of relatively hyperbolic groups. Then, in order to follow Delzant's idea in [Del], we will generalize, in section 2, the construction of canonical cylinders of Rips and Sela [RS] (Theorems 2.9 and 2.22). The main difficulty comes from the fact that the considered hyperbolic graph is no longer locally finite. Finally, we prove Theorem 0.2 in section 3.

ACKNOWLEDGEMENTS: I would like to thank Thomas Delzant for interesting discussions on the subject. I am deeply grateful to Brian Bowditch for his useful comments on this work. I am also very grateful to the referee.

1. Relatively hyperbolic groups

1.1 DEFINITIONS. A graph is a set of vertices with a set of edges, which are pairs of vertices. One can equip a graph with a metric where edges have length 1. Thus this geometrical realization allows one to consider geodesic, quasi-geodesic and locally geodesic paths in a graph. A circuit in a graph is a simple simplicial loop (without self-intersection). In [B2], B. Bowditch introduces fine graphs:

Definition 1.1 (Fineness)[B2]*:* A graph $\mathcal{K}$ is **fine** if for all $L > 0$, for every edge e, the set of the circuits of length less than L, containing e, is finite. It is uniformly fine if this set has cardinality bounded above by a constant depending only on L.

We will use this definition as a finiteness property of certain non-locally finite graphs.

Definition 1.2 (Relatively Hyperbolic Groups)[B2]*:* A group Γ is **hyperbolic relative to a family of subgroups** $\mathcal{G}$, if it acts on a hyperbolic and fine graph $\mathcal{K}$, such that stabilizers of edges are finite, the quotient $\Gamma\backslash\mathcal{K}$ is a finite graph, and the stabilizers of the vertices of infinite valence are exactly the elements of $\mathcal{G}$, and are finitely generated.

We will say that such a graph is **associated** to the relatively hyperbolic group Γ. A subgroup of an element of $\mathcal{G}$ is said to be parabolic.

As there are finitely many orbits of edges, a graph associated to a relatively hyperbolic group is uniformly fine.

1.2 ANGLES AND CONES. As already explained in [DY], from which this section is partially borrowed, angles and cones are useful tools for the study of fine graphs.

Definition 1.3 (Angles): Let $\mathcal{K}$ be a graph, and let $e_1 = (v, v_1)$ and $e_2 = (v, v_2)$ be edges with one common vertex v. The angle $\text{Ang}_v(e_1, e_2)$ is the shortest length of the paths from v_1 to v_2, in $\mathcal{K} \setminus \{v\}$. It is $+\infty$ if there is no such path.

The angle $\text{Ang}_v(p, p')$ between two simple simplicial (oriented) paths p and p' having a common vertex v is the angle between their first edges after this vertex.

If p is a simple simplicial path, and v one of its vertices, $\text{Ang}_v(p)$ is the angle between the consecutive edges of p at v, and its maximal angle $\text{MaxAng}(p)$ is the maximal angle between consecutive edges of p.

In the notation $\text{Ang}_v(p, p')$, we will sometimes omit the subscript if there is no ambiguity.

PROPOSITION 1.4 (Three useful remarks):

1. *When defined:* $\text{Ang}_v(e_1, e_3) \leq \text{Ang}_v(e_1, e_2) + \text{Ang}_v(e_2, e_3)$.
2. *If γ is an isometry,* $\text{Ang}_v(e_1, e_2) = \text{Ang}_{\gamma v}(\gamma e_1, \gamma e_2)$.
3. *Any circuit (simple loop) of length $L \geq 2$ has a maximal angle less than $L - 2$.*

Proof: The first statement follows from the triangular inequality for the length distance of $\mathcal{K} \setminus \{v\}$. The second statement is obvious. Finally, if $e_1 = (v_1, v)$ and $e_2 = (v, v_2)$ are two consecutive edges in the circuit, the circuit itself gives a path of length $L - 2$ from v_1 and v_2 avoiding v. ∎

LEMMA 1.5 (Large angles in triangles): *Let $[x, y]$ and $[x, z]$ be geodesic segments in a δ-hyperbolic graph, and assume that* $\text{Ang}_x([x, y], [x, z]) = \theta \geq 50\delta$. *Then the concatenation of the two segments is still a geodesic. Moreover, x belongs to any geodesic segment $[y, z]$ and* $\text{Ang}_x([y, z]) \geq \theta - 50\delta$.

Proof: Let $[y, z]$ be a geodesic, defining a triangle (x, y, z), which is δ-thin. If both segments $[x, y]$ and $[x, z]$ are shorter than 10δ, then, $|y - z| \leq 20\delta$, and the total length of the edges of the triangle is less than 40δ. The third part of Proposition 1.4 proves that $x \in [y, z]$, and $Ang_x([y, z]) \geq \theta - 50\delta$.

Assume that $[x, y]$ is shorter than 10δ, and that $[x, z]$ is longer than 10δ. Let z' be the point on $[x, z]$ at distance 13δ from x. By triangular inequality, $|z' - y| \geq 3\delta$, and therefore there exists a vertex z'' on $[y, z]$ at distance at most δ from z'. By triangular inequality, the segment $[z'', y]$ has length at most 24δ, and therefore the loop $[x, z'][z', z''][z'', y][y, x]$ has length at most 48δ. The segment $[z', z'']$ is at distance at least 12δ from x, and therefore it does not contain it. Again, the third part of Proposition 1.4 proves that $x \in [y'', z'']$, and $\mathrm{Ang}_x([y'', z'']) \geq \theta - 50\delta$.

Assume now that both $[x, y]$ and $[x, z]$ are longer than 10δ. We consider the vertices y' and z' on $[x, y]$ and $[x, z]$ located at distance 10δ from x. If there is a path of length less than 3δ between y' and z', it cannot contain x, and therefore it would contradict that the angle at x is greater than 50δ. Therefore y' and z' are not 3δ-close to each other, thus they are δ-close to the segment $[y, z]$, and we set y'', respectively z'', in $[y, z]$ at distance less than δ from x', respectively y'. Consider the loop $[x, y'][y', y''][y'', z''][z'', z'][z', x]$. Its length is less than $(2 \times 10\delta + 2\delta) \times 2 \leq 50\delta$, and it contains x. The segments $[y', y'']$ and $[z', z'']$ are at distance at least 9δ from x, so that they do not contain x. Here again, the third part of Proposition 1.4 proves that $x \in [y'', z'']$, and $\mathrm{Ang}_x([y'', z'']) \geq \theta - 50\delta$. ∎

Definition 1.6 (Cones): Let $\mathcal{K}$ be a graph, $d > 0$ and $\theta > 0$. Let e be an edge, and v one of its vertices. The **cone** centered at (e, v), of radius d and angle θ, is the set of vertices w at distance less than d from v and such that there exists a geodesic $[v, w]$ satisfying the property that its maximal angle and its angle with e are less than θ:

$$Cone_{d,\theta}(e, v) = \{w \,|\, |w - v| \leq d, \mathrm{MaxAng}[v, w] \leq \theta, \mathrm{Ang}_v(e, [v, w]) \leq \theta\}.$$

PROPOSITION 1.7 (Bounded angles imply local finiteness): *Let $\mathcal{K}$ be a fine graph. Given an edge e and a number $\theta > 0$, there exist only finitely many edges e' adjacent to e such that* $\mathrm{Ang}(e, e') \leq \theta$.

Proof: There are only finitely many circuits shorter than θ containing e. ∎

COROLLARY 1.8 (Cones are finite): *In a fine graph, the cones are finite sets. If the graph is uniformly fine, the cardinality of $Cone_{d,\theta}(e, v)$ can be bounded above by a function of d and θ.*

Proof: Consider a cone $Cone_{d,\theta}(e, v)$. We argue by induction on d. If $d = 1$, the result is given by the previous proposition. If $d > 1$, we remark that $Cone_{d,\theta}(e, v)$ is contained in the union of cones of angle θ and radius 1, centered at edges whose vertices are both in $Cone_{(d-1),\theta}(e, v)$. If the latter is finite, the union is also finite. ∎

LEMMA 1.9 (Cones and circuits): *Let e be an edge of a graph, and w a vertex that lies in a circuit containing e and of length less than L. Then $w \in Cone_{L,L}(e,v)$.*

Proof: Let C be the considered circuit, and let g be a geodesic segment between v and w. The concatenation of g and one of the two paths in C from w to v is a loop. Hence, one has two loops containing g, one of them containing e, one not, and both of length less than L. If g has an angle greater than L, then the corresponding vertex would not be in a sub-circuit of each of the two loops, and therefore the circuit C would pass through this point twice, which contradicts the definition of circuit. For the same reason the angle between e and g is less than L, and therefore $w \in Cone_{L,L}(e,v)$. ∎

Definition 1.10: Let Λ be a number. A Λ-quasi-geodesic in a metric space X is a path q: $[a,b] \to X$ such that for all x and y, $|x-y|/\Lambda \leq \mathrm{dist}(q(x),q(y)) \leq \Lambda|x-y|$.

PROPOSITION 1.11 (Conical stability of quasi-geodesics): *Let Λ be a positive number. In a δ-hyperbolic graph $\mathcal{K}$, let g: $[a,b] \to \mathcal{K}$ be a geodesic segment, and let q: $[a,b] \to \mathcal{K}$ be a Λ-quasi-geodesic with $|q(a)-g(a)| \leq r$ and $|q(b)-g(b)| \leq r$, for $r \leq 10\delta$. Let w be a vertex in q at distance at least $2r$ from the ends. Then there exists a constant $N_{\Lambda,\delta}$ depending only on Λ, and δ, and there exists an edge e in g, such that $w \in Cone_{N_{\Lambda,\delta},N_{\Lambda,\delta}}(e,v)$.*

Proof: It is a classical fact ([G], 7.2 A, [CDP], [GH]) that there exists a number $D(\Lambda,\delta)$ such that q remains at a distance less than $D(\Lambda,\delta)$ from the segment. We consider the loop starting at w, consisting of five parts: a subsegment $[w,w_1]$ of q, of length less than $10D(\Lambda,\delta)$, and strictly less if and only if $w_1 = q(b)$; a segment $[w_1,w_2]$ of length less than $D(\Lambda,\delta)$ and where $w_2 \in g$ (we call it a transition); a subsegment $[w_2,w_3]$ of g of length less than $20D(\Lambda,\delta)$ (strictly less if and only if $w_3 = g(a)$); then again a transition from w_3 to q shorter than $D(\Lambda,\delta)$; and then a subsegment of q to w. As in any case w is sufficiently far from the transitions, with respect to their length, it does not belong to them, and this loop contains a sub-circuit shorter than $25\Lambda D(\Lambda,\delta)$, containing w and an edge of g. Lemma 1.9 gives the result. ∎

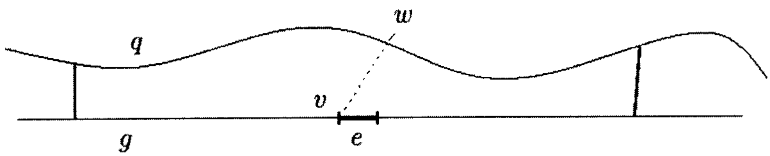


Figure 1. Quasi-geodesics stay in cones centered on the geodesic, Proposition 1.11.

2. Canonical cylinders for a family of triangles

In the following, $\mathcal{K}$ is a graph associated to a relatively hyperbolic group Γ, and is δ-hyperbolic. We choose a base point p in $\mathcal{K}$. We assume, without loss of generality, that δ is an integer greater than 1.

The aim of this section is, given a finite family F of elements of Γ, to find a finite set (a **cylinder**) around each segment $[p, \gamma p]$ with $\gamma \in F \cup F^{-1}$. This construction will be such that for all α, β, γ in $F \cup F^{-1}$ which satisfy the equation ($\alpha\beta\gamma = 1$), the three cylinders around $[p, \alpha p]$, $[\alpha p, \alpha\beta p] = \alpha[p, \beta p]$ and $[p, \gamma^{-1}p] = [\alpha\beta\gamma p, \alpha\beta p]$ coincide pairwise on large subsets around the vertices p, αp and $\alpha\beta p$ (see Theorem 2.9).

Our approach is similar to the original one in [RS]. However, let us emphasize that Rips and Sela use the fact that the balls in Cayley graphs are finite. In the graph we are working on, it is not the case.

2.1 Coarse piecewise geodesics. We choose some constants: $\lambda = 1000\delta$, $\epsilon = N_{\lambda,\delta}$ and $\mu = (100\epsilon + \lambda^2) \times 40\lambda$ ($N_{\lambda,\delta}$ is as in Proposition 1.11). These constants will be useful for defining **coarse piecewise geodesics**, in the sense of [RS]. Roughly speaking, λ will serve as constant for quasi-geodesics, μ will serve as constant for local geodesics, and ϵ will be the bound for lengths of **bridges**.

A path p is a μ-local-geodesic if any subpath of p of length μ is a geodesic. A path p is a L-local-$\lambda/2$-quasi-geodesic if any subpath of length at most L is a $\lambda/2$-quasi-geodesic.

Definition 2.1 (Coarse piecewise geodesics)([RS] 2.1): Let l be an integer greater than μ. A l-coarse-piecewise-geodesic in $\mathcal{K}$ is a $40\lambda(\epsilon + 100\lambda\delta)$-local-$\lambda/2$-quasi-geodesic f: $[a, b] \to \mathcal{K}$ together with a subdivision of the segment $[a, b]$, $a = c_1 \leq d_1 \leq c_2 \leq \cdots \leq d_n = b$ such that

- $f([c_i, d_i])$ is a μ-local geodesic,
- $\forall i, 2 \leq i \leq (n-1), \mathit{length}(f([c_i, d_i])) \geq l$ and $\forall i, \mathit{length}(f[d_i, c_{i+1}]) \leq \epsilon$,
- $f([a, b])$ is included in the 2ϵ-neighborhood of a geodesic segment $[f(a), f(b)]$.

In this case, we say that $f|_{[c_i,d_i]}$ is a sub-local-geodesic, and $f|_{[d_i,c_{i+1}]}$ is a bridge.

Remark 1: Any l-coarse-piecewise-geodesic is a λ-quasi-geodesic. This follows from [G] 7.2B, where it is stated that any $1000\delta\frac{\lambda}{2}$-local-$\lambda/2$-quasi-geodesic is a λ-quasi-geodesic. We also give, in the appendix of this paper, a simple proof that uses the third point of the definition.

Remark 2: If $f\colon [a, b] \to \mathcal{K}$ is a coarse-piecewise-geodesic, then for all a' and b' such that $a \leq a' < b' \leq b$, the path $f_{[a',b']}$ is a coarse-piecewise-geodesic. Indeed, the induced subdivision satisfies the two first points of the definition (note that there is no length condition for the first and the last sub-local geodesic), and the third point is satisfied since f (and therefore $f|_{[a',b']}$) is a λ-quasi-geodesic, and Proposition 1.11 applies.

LEMMA 2.2 (Re-routing coarse piecewise geodesics): *Let $l \geq \mu$ be a number, and $f\colon [a, b] \to \mathcal{K}$ be a l-coarse-piecewise-geodesic. Consider a sub-local-geodesic $f|_{[c,d]}$, and $s \in f([c, d])$, with the additional requirement that the subpath of $f([c, d])$ from $f(c)$ to s has length more than $l + 2\epsilon$. Let g be a geodesic segment between $f(a)$ and $f(b)$. Let s'' be a closest point to s on g. Let s' be a closest point to s'' on $f_{[c,d]}$.*

We choose $[s', s'']$ a geodesic segment between s' and s''. We note $[s'', f(b)]$ is a subsegment of g between s'' and $f(b)$. We note $(f(a), s')$ is the image by f of the real segment $[a, f^{-1}(s')]$.

Then the path $\tilde{f} = (f(a), s')[s', s''][s'', f(b)]$ is a l-coarse-piecewise-geodesic. We say that f is re-routed into $\tilde{f}$.

Proof: We define the parameterization of $\tilde{f}$ on some real segment $[a, b']$, for some $b' = f^{-1}(s') + \mathit{length}([s', s''][s'', f(b)])$, to coincide with the one of f on $[a, f^{-1}(s')]$, and to be the arc length parametrization on $[f^{-1}(s'), b']$. Let $a = c_1 < d_1 \leq c_2 < \cdots \leq c_m < d_m = b$ be the subdivision of $[a, b]$ associated to f, and let n be such that $c = c_n$, with c as in the statement. We define the subdivision of $[a, b']$, $a = c'_1 < d_1 \leq c'_2 \leq \cdots \leq c'_n < d'_n \leq c'_{n+1} < d'_{n+1} = b'$ as coinciding with the one of the coarse-piecewise-geodesic f until $c'_n = c = c_n$, and such that $d'_n = \tilde{f}^{-1}(s')$, $c'_{n+1} = \tilde{f}^{-1}(s'')$, and $d_{n+1} = b'$. It is immediate from a similar property for f that any restriction $f|_{[c_i,d_i]}$ is a μ-local geodesic, that for all $i \in [2, n-1]$, $length(f([c_i, d_i])) \geq l$, and that for all $i \in [1, n-1]$, $length(f([d_i, c_{i+1}])) \leq \epsilon$. We know that f is a λ-quasi-geodesic, therefore by 1.11, it stays ϵ-close to g, hence, by triangular inequality, $\tilde{f}$ also stays ϵ-close to g.

We have to show that $length(f([d_n, c_{n+1}])) \leq \epsilon$, that $length(f([c_n, d_n])) \geq l$ and that $\tilde{f}$ is a $40\lambda(\epsilon + 100\lambda\delta)$-local-$\lambda/2$-quasi-geodesic.

As $|s - s''| \leq \epsilon$, the segment $[s', s'']$ has length less than ϵ, which was the first requirement.

Therefore, $|s - s'| \leq 2\epsilon$, and, as we assumed that the length of f from $f(c)$ to s is greater than $l + 2\epsilon$, the sub-local-geodesic of $\tilde{f}$ between $f(c)$ and s' is longer than l in this case, which was the second requirement.

We need to prove that $\tilde{f}$ is a $40\lambda(\epsilon+100\lambda\delta)$-local-$\lambda/2$-quasi-geodesic. In other words, we have to show that any of its subpath of length less than $40\lambda(\epsilon+100\lambda\delta)$ is a $\lambda/2$-quasi-geodesic. Let p be such a subpath. If it is contained in the subpath of $\tilde{f}$ coinciding with $f([a, b])$, then by assumption on f, it is a $\lambda/2$-quasi-geodesic. If it is contained in the subpath of $\tilde{f}$ coinciding with g, then it is a geodesic segment. If p fails to satisfy both conditions above, then it intersects ρ, and therefore is contained in a subpath of length at most $40\lambda(\epsilon + 100\lambda\delta) + 2\epsilon$ that contains ρ. We give some notation: let x and y be the ends of this subpath. As $\mu \geq 40\lambda(\epsilon + 100\lambda\delta) + 4\epsilon$, the subsegment $[x, s']$ of f is a geodesic. The segment $[s'', f(b)]$ is included in g and therefore it is a geodesic segment, and it contains y. If the length of p is less than $\lambda/2 = 500\delta$, there is nothing to prove.

It is now enough to prove that for every such subpath p containing ρ, of length more than 500δ, the distance $|x - y|$ between the ends x and y of p is greater than $\frac{1}{\lambda}(|x - s'| + |s' - s''| + |s'' - y|)$.

As the point s' is the closest point to s'' in $[x, s]$, by hyperbolicity, we have $|x - s'| + |s' - s''| \leq |x - s''| + 5\delta$.

Consider a point u of the sub-local-geodesic $f([c, d])$ that is between $f(c)$ and x, and at distance $\mu/2$ from x. As $l \geq \mu \geq 40\lambda(\epsilon + 100\lambda\delta) + 2\epsilon \geq |x - s'|$, it is possible to find such a point. Note that the subpath $[u, s']$ of f is of length at most μ and therefore is a geodesic segment. Moreover, by Proposition 1.11, there is a point v on g such that $|u - v| \leq \epsilon$. As the Gromov product $(v \cdot y)_{s''}$ is equal to zero, and as $(v \cdot u)_{s''} \geq \mu - 2\epsilon - 10\delta \geq 100\delta$, by hyperbolicity, one has $(y \cdot u)_{s''} \leq \delta$.

Similarly, $(u \cdot s'')_x \leq 2\delta$, that is $(u \cdot x)_{s''} \geq |s'' - x| - 5\delta$. There is the dichotomy: either $|s'' - x| \leq 20\delta$, hence $|s'' - y| \geq length(p) - 25\delta \geq |y - x| - 25\delta$, and

$(y \cdot x)_{s''} \leq 45\delta$, or $|s'' - x| \geq 20\delta$, and then $(u \cdot x)_{s''} \geq 20\delta$, which together with $(y \cdot u)_{s''} \leq \delta$ yields $(y \cdot x)_{s''} \leq 2\delta$. In any case, one has $(y \cdot x)_{s''} \leq 45\delta$. Then $|x - y| \geq |x - s''| + |s'' - y| - 45\delta$. We already had $|x - s'| + |s' - s''| \leq |x - s''| + 5\delta$, which gives $|x - y| \geq |x - s'| + |s' - s''| + |s'' - y| - 50\delta$, and as $|x - s'| + |s' - s''| + |s'' - y|$ was assumed to be greater than 500δ, this gives the expected $|x - y| \geq \frac{1}{\lambda} \times (|x - s'| + |s' - s''| + |s'' - y|)$. This proves the proposition. ∎

LEMMA 2.3: *Let $[x, y]$ be a geodesic segment of $\mathcal{K}$, of length $L \geq 2\mu$. Let s be on $[x, y]$ such that $|s - x|$ and $|s - y|$ are both greater than $\mu/2$. Let $s' \in \mathcal{K}$ be at distance at most δ from s and $y' \in \mathcal{K}$ be at distance at most δ from y. Let s'' be on $[x, y]$ such that $|s' - s''|$ is minimal, and let $[s', s'']$ be any geodesic segment. Then the path $[x, s''][s'', s'][s', y']$ is a $40\lambda(\epsilon + 100\lambda\delta)$-local-$\lambda/2$-quasi-geodesic.*

Proof: As in the previous lemma, it is enough to prove that for every subpath p containing $[s'', s']$, of length more than $500\delta = \lambda/2$, the distance between the ends p_1 and p_2 of p is greater than $\frac{1}{\lambda} \times (|p_1 - s'| + |s' - s''| + |s'' - p_2|)$.

Let us assume that $|s'' - p_2| \geq 25\delta$. By hyperbolicity, p_2 is 5δ-close to a point w of $[s', y]$, and $|s' - w| \geq |s'' - p_2| - |p_2 - w| - |s' - s''|$. Now

$$|p_1 - w| = |p_1 - s'| + |s' - w| \geq |p_1 - s'| + |s'' - p_2| - |p_2 - w| - |s' - s''|.$$

As $|p_1 - p_2| \geq |p_1 - w| - |w - p_2|$ we deduce that

$$\begin{aligned} |p_1 - p_2| &\geq |p_1 - s'| + |s'' - p_2| - 2|p_2 - w| - |s' - s''| \\ &\geq |p_1 - s'| + |s'' - p_2| + |s' - s''| - 12\delta, \end{aligned}$$

which is greater than $\frac{1}{1000\delta} \times (|p_1 - s'| + |s' - s''| + |s'' - p_2|)$, since $|p_1 - s'| + |s' - s''| + |s'' - p_2|$ is assumed to be greater than 500δ.

If $|s'' - p_2| \leq 25\delta$, then

$$|p_1 - p_2| \geq |p_1 - s''| - |s'' - p_2| \geq |p_1 - s'| + |s' - s''| + |s'' - p_2| - 51\delta,$$

and the same conclusion holds. ∎

COROLLARY 2.4 (Re-routing to another point): *Let l be a positive number. Let $f\colon [a, b] \to \mathcal{K}$ be a coarse-piecewise-geodesic whose last sub-local-geodesic g is a geodesic segment of length at least $l + 2\mu$. Let $[f(a), f(b)]$ be a geodesic segment of $\mathcal{K}$, and let z be a point such that a geodesic segment $[f(a), z]$ passes* *at distance δ from $f(b)$. Then, there exists a l-coarse-piecewise-geodesic from $f(a)$ to z coinciding with f until the first point of g.*

Proof: Let $f\colon [c,b] \to \mathcal{K}$ be an arc-length parametrization of the sub-local-geodesic g. One has $(b-c) \geq l + 2\mu$. Let $x = f(c+l)$ and $y = f(b)$. Let y' be a point of $[f(a), z]$ at distance less than δ from y.

By the previous lemma, there exist points s'' on g, s' on $[f(a), y']$ such that the path $f([a,c])[x,s''][s'',s'][s',y']$ is a $40\lambda(\epsilon + 100\lambda\delta)$-local-$\lambda/2$-quasi-geodesic satisfying the two first points of the definition of l-coarse-piecewise-geodesic. As $[s', y']$ is a subsegment of the geodesic segment $[s', z]$, the same is true for $f([a,c])[x,s''][s'',s'][s',y']$.

It remains to show that this path stays 2ϵ-close to a geodesic segment $[f(a), z]$. Its first part from $f(a)$ to s'' is a subpath of f, hence it is a λ-quasi-geodesic, therefore ϵ-close to $[f(a), f(b)]$, and therefore, $(\epsilon + \delta)$-close to $[f(a), z]$. The second part $[s'', s'][s', z]$ is ϵ-close to $[s', z] \subset [f(a), z]$ since $|s'' - s'| \leq \epsilon$. This proves the claim. ∎

2.2 Cylinders. We now define the **cylinders**, which are subsets of $\mathcal{K}$ associated to pairs of points.

Definition 2.5 (l-Cylinders)[RS]*:* Let $l \in \mathbb{N}$. The l-cylinder of two points x and y in $\mathcal{K}$, denoted by $Cyl_l(x,y)$, is the set of the vertices v lying on a l-coarse-piecewise-geodesic from x to y, with the additional requirement that v is on a sub-local-geodesic $f|_{[c,d]}$ with distances $|f(c) - v| \geq l$ if $f(c) \neq x$ and $|f(d) - v| \geq l$ if $f(d) \neq y$.

The next lemma will assure that cylinders are finite sets, and stay close to geodesics.

Lemma 2.6 (Cylinders are finite): *Given two points x and y in $\mathcal{K}$, and a constant l, any l-coarse-piecewise-geodesic from x to y remains in the union of the cones of radius and angle ϵ centered in the edges of an arbitrary geodesic segment $[x,y]$ (we call this union the ϵ-conical-neighborhood of the segment). In other words, the l-cylinder $Cyl_l(x,y)$ is contained in the union of the cones of radius and angle ϵ-conical-neighborhood of an arbitrary geodesic segment $[x,y]$.*

The l-cylinder $Cyl_l(x,y)$ contains every geodesic between x and y.

Proof: The first assertion is a consequence of Proposition 1.11 for $\Lambda = \lambda$, and $r = 0$. To prove the last assertion it is sufficient to remark that every geodesic is a l-coarse-piecewise-geodesic with only one sublocal geodesic, and no bridge. ∎

LEMMA 2.7 (Equivariance): *If a vertex v is in $Cyl_l(x,y)$, then for all γ in the group Γ, we have $\gamma v \in Cyl_l(\gamma x, \gamma y)$.*

Proof: Multiplication on the left by γ is an isometry of $\mathcal{K}$. ∎

2.3 CHOOSING A GOOD CONSTANT l FOR l-CYLINDERS.

Definition 2.8 (Channels)([RS] 4.1): Let $g = [v_1, v_2]$ be a geodesic segment in $\mathcal{K}$. A geodesic not shorter than $|v_2 - v_1|$ that stays in the union of the cones of radius and angle ϵ centered in the edges of g is a ($|v_2 - v_1|$)-**channel** of g.

As cones are finite (Corollary 1.8), the number of different channels of a segment of length L is bounded above by a constant depending only on L. We note $Capa(L)$ is such a bound.

Let us recall the constants we fixed, and that are involved in the definition of coarse piecewise geodesics: $\mu = (100N_{\lambda,\delta} + \lambda^2) \times 40\lambda$, with $\lambda = 1000\delta$. For an integer n, we set $\varphi(n) = 24(n+1)Capa(\mu)(2\epsilon + 1)\epsilon$. For $1 \leq i \leq \varphi(n)/2\epsilon$, we set now $l_i = 10\mu + 2i\epsilon$. Each l_i is inferior to $\varphi(n) + 10\mu$.

We denote by $B_r(x)$ the ball of $\mathcal{K}$ of center x and radius r.

THEOREM 2.9: *Let F be a finite family of elements of Γ; we set $n = (2\,\mathrm{Card}(F))^3$ where $\mathrm{Card}(F)$ is the cardinality of F. Let p be a base point in $\mathcal{K}$.*

There exists a number l such that the l-cylinders satisfy: for all α, β, γ in $F \cup F^{-1}$ with $\alpha\beta\gamma = 1$, in the triangle $(x,y,z) = (p, \alpha p, \gamma^{-1}p)$ in $\mathcal{K}$, one has

$$Cyl_l(x,y) \cap B_{R_{x,y,z}}(x) = Cyl_l(x,z) \cap B_{R_{x,y,z}}(x)$$

(and analogues permuting x, y and z), where $R_{x,y,z} = (y \cdot z)_x - 4 \times (12\mu + \varphi(n))$ is the Gromov product in the triangle, minus a constant.

What is important in the theorem is not so much the value of l, but that the numbers $(y \cdot z)_x - R_{x,y,z}$ involved are bounded in terms of n and of $\mathcal{K}$ (namely, δ and the cardinality of a cone of radius and angle ϵ). This bound does not depend on the family F, although it does depend on its cardinality.

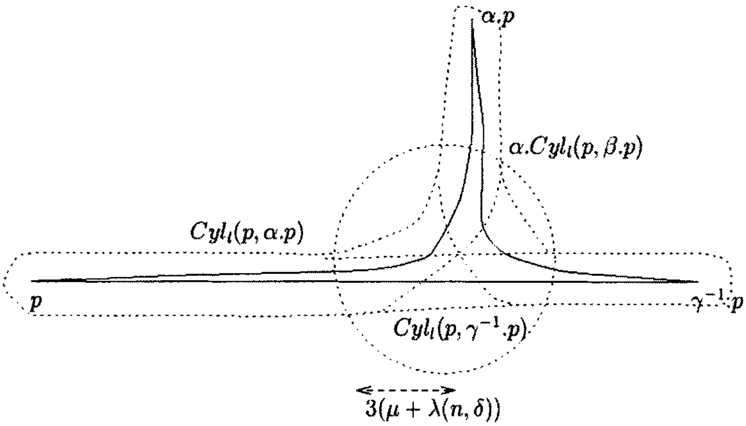


Figure 2. Cylinders for a triangle, Theorem 2.9.

Proof: We will find a correct constant l among the l_i previously defined. We have $12(n+1)Capa(\mu)(2\epsilon+1)$ different candidates. There are at most n different triangles satisfying the condition, hence we have a system of at most $6n$ equations of the form $Cyl_l(x,y)\cap B_{R_{x,y,z}}(x)\subset Cyl_l(x,z)\cap B_{R_{x,y,z}}(x)$. It is then enough to prove the next lemma.

LEMMA 2.10: *Let x,y,z be three points in $\mathcal{K}$. There are at most $2Capa(\mu)(2\epsilon+1)$ different constants among the l_i, $i=1,\dots,\varphi(n)/2\epsilon$, such that $Cyl_l(x,y)\cap B_{R_{x,y,z}}(x)\not\subset Cyl_l(x,z)\cap B_{R_{x,y,z}}(x)$.*

Proof: We argue by contradiction, assuming that $(2Capa(\mu)(2\epsilon+1)+1)$ constants l_i do not satisfy the equation

$$Cyl_l(x,y)\cap B_{R_{x,y,z}}(x)\subset Cyl_l(x,z)\cap B_{R_{x,y,z}}(x).$$

For each of them, there is a vertex v_i in one cylinder and not in the other: there exists β_i, a l_i-coarse-piecewise-geodesic from x to y containing v_i as indicated in Definition 2.5, but there is no such coarse-piecewise-geodesic from x to z.

By Lemma 2.6, given an arbitrary geodesic segment $[x,y]$, each of the coarse-local-geodesics β_i $(i=1,\dots,\varphi(n)/2\epsilon)$ is contained in the ϵ-conical-neighborhood (in the sense of Lemma 2.6) of $[x,y]$. Thus, every subsegment of length μ of a sub-local-geodesic of β_i is a μ-channel of a subsegment of length μ of $[x,y]$.

Let $[x_1', x_1'']$ and $[x_2', x_2'']$ be two subsegments of $[x, y]$ of length μ, and such that $|x_1' - x| = R_{x,y,z} + (\varphi(n) + 10\mu)$ and $|x_2' - x| = R_{x,y,z} + (\varphi(n) + 12\mu)$. As $\epsilon \leq \mu/10$, and $l_i \geq 10\mu$, each of the β_i has a sub-local-geodesic passing through a μ-channel of *either* $[x_1', x_1'']$, *or* $[x_2', x_2'']$. Indeed, if it fails to have one in the first case, β_i must have a bridge at distance $\mu + \epsilon$ from x_1', and therefore no bridge at distance $\mu + 5\epsilon$ from x_2'', hence the second case holds.

There are at most $2Capa(\mu)$ such channels. Therefore, there is a channel, denoted by $Chan$, in which a sub-local geodesic $\beta_i' = \beta_i|_{[c_i, d_i]}$ passes for at least $2\epsilon + 2$ different indexes $i \in [1, \varphi(n)/2\epsilon]$. Let us label $2\epsilon + 2$ of these indexes: $i_1 < i_2 < \cdots < i_{2\epsilon+2}$.

For each $1 \leq j \leq 2\epsilon + 2$, let $t_j \in [c_{i_j}, d_{i_j}]$ be the instant where $\beta_{i_j}'(t_j)$ exits the channel $Chan$. Let us denote by $r(\beta_{i_j}')$ the length of the path $\beta_{i_j}'([t_j, d_{i_j}])$, the part of β_{i_j}' after it leaves the channel $Chan$. The discussion will hold on the respective possible values of the numbers $r(\beta_{i_j}')$, for $1 \leq j \leq 2\epsilon + 2$.

We now formulate and prove three claims.

Claim 1: *For any $j \in [1, 2\epsilon + 2]$, one has $r(\beta_{i_j}') \leq l_{i_j} + 2\epsilon$.*

Assume the contrary. Let $t_j^+ > t_j$ be a real number such that the length of $\beta_{i_j}'([t_j, t_j^+])$ equals l_{i_j}. Then, by Lemma 2.2, β_{i_j} can be rerouted into a l_i-coarse-piecewise-geodesic coinciding with β_{i_j} from x to $\beta_{i_j}(t_j^+)$, coinciding with $[x, y]$ on a suffix starting at a point 3ϵ-close to $\beta_{i_j}(t_j^+)$, and ending at y. Recall that $\beta_{i_j}(t_j)$ is the end point of the channel $Chan$. By triangular inequality, $|x - \beta_{i_j}(t_j)| \leq R_{x,y,z} + (\varphi(n) + 12\mu) + \epsilon + \mu$. Therefore, $\beta_{i_j}(t_j^+)$ is at distance at most $[R_{x,y,z} + (\varphi(n) + 12\mu) + \epsilon + \mu] + (\varphi(n) + 10\mu) + \epsilon \leq (y \cdot z)_x - 2\varphi(n) - 22\mu$ from x. Therefore, by Corollary 2.4, it can be rerouted, at distance $l_{i_j} \leq (\varphi(n) + 10\mu)$ after the bridge of the first re-routing, into a l_i-coarse-piecewise-geodesic coinciding with β_{i_j} from x to $\beta_{i_j}(t_j^+)$ that ends at z. This shows that v_{i_1} is in the cylinders $Cyl_{i_1}(x, z)$, which contradicts our assumption, and proves the claim.

Claim 2: *For any two indices $i_j < i_k$, one has $r(\beta_{i_k}') < r(\beta_{i_j}')$.*

If not, one could change β_{i_j} just after $Chan$, into β_{i_k} (it remains a l_{i_j}-coarse-piecewise-geodesic). On β_{i_k}, let β_{i_k}'' be the sublocal geodesic following β_{i_k}'. It is of length $l_{i_k} \geq l_{i_j} + 2\epsilon$. Let t_k^+ be real such that $\beta_{i_k}(t_k^+)$ is the point located on β_{i_k}'' at distance $l_{i_j} + 2\epsilon$ from its beginning. Recall (see paragraph above) that $|x - \beta_{i_j}(t_j)| \leq [R_{x,y,z} + (\varphi(n) + 12\mu) + \epsilon + \mu]$. By Claim 1, we deduce that $\beta_{i_k}(t_k^+)$ is at distance at most $[R_{x,y,z} + (\varphi(n) + 12\mu) + \epsilon + \mu] + 2 \times (\varphi(n) + 10\mu) + 3\epsilon \leq (y \cdot z)_x - 2 \times (\varphi(n) + 10\mu + 2\epsilon)$ from x, and therefore, by Lemma 2.2, it is possible to reroute the coarse-piecewise-geodesic, through a l_{i_j}-coarse-piecewise-geodesic coinciding with β_{i_j} until $Chan$, and coinciding with the suffix of $[x, y]$ starting at $(y \cdot z)_x - 2 \times (\varphi(n) + 10\mu)$ from x, and ending at y. Then, by Corollary 2.4, it is possible to reroute it again through a l_{i_j}-coarse-piecewise-geodesic ending at z, which is a contradiction, as in Claim 1.

CLAIM 3: *For all* $i_k \leq 2\epsilon + 2$, *one has* $r(\beta'_{i_1}) - r(\beta'_{i_k}) < 2\epsilon$.

If not, we could change β_{i_k} just after $Chan$, by passing by β'_{i_1}, and reroute it on $[x, y]$ before the end of β'_{i_1} (at distance 2ϵ from the end). This again gives the same contradiction.

Now that the three claims have been proved, we can end the proof of Lemma 2.10.

We see from the second claim that the $2\epsilon + 2$ numbers $r(\beta'_{i_j})$, for $j \leq 2\epsilon + 2$, are all different, and, from the third claim, that they are all in an interval of $\mathbb{N}$ of length 2ϵ (hence containing $2\epsilon + 1$ elements). This is a contradiction. ∎

2.4 DECOMPOSITION OF CYLINDERS INTO SLICES. In this section we assume that the hypotheses of Theorem 2.9 are fulfilled, and we choose l a suitable constant as in the statement of this theorem. All considered cylinders will implicitly be l-cylinders.

Recall that there are only finitely many orbits of vertices of finite valence in $\mathcal{K}$, therefore there exists a constant ρ such that any pair of edges (e, e') containing a vertex of finite valence v makes an angle $\mathrm{Ang}_v(e, e')$ bounded above by ρ.

Let $\Theta = \max\{10000(D + \epsilon + \delta), \rho\}$, where D is a constant such that any λ-quasi-geodesic remains at distance D from any geodesic in a δ-hyperbolic graph (here $\lambda = 1000\delta$).

The decomposition into slices given by Rips and Sela in the hyperbolic case ([RS]) will not work properly here, because of large angles. Thus, we choose a slightly different procedure.

Definition 2.11 (Parabolic slices in a cylinder): Let a and b be two points in $\mathcal{K}$. In the cylinder $Cyl(a, b)$, a **parabolic slice** is a singleton $\{v\} \subset Cyl(a, b)$ such that there exist vertices w and w' in $Cyl(a, b)$, adjacent to v in $\mathcal{K}$ and such that $\mathrm{Ang}_v((v, w), (v, w')) \geq \Theta$. The angle of a parabolic slice is $Max_{w,w' \in Cyl(a,b)}(\mathrm{Ang}_v((v, w), (v, w')))$.

Note that, since $\Theta \geq \rho$, any parabolic slice consists in a vertex of infinite valence. This justifies the name. A non-parabolic slice can contain a single vertex of infinite valence.

LEMMA 2.12 (Parabolic slice implies large angle on a geodesic segment): *Let a and b be two points in $\mathcal{K}$, and let A be a number greater than Θ. If w and w' are vertices in the cylinder $Cyl(a,b)$, such that $|w-w'| \leq 50\delta$, and if there exists v on some geodesic $[w,w']$ such that $\mathrm{Ang}_v([w,w']) = A$, then any geodesic segment $[a,b]$ contains v, and $\mathrm{Ang}_v([a,b]) \geq A - 20D \geq A - \Theta$.*

If $\{v\}$ is a parabolic slice of a cylinder $Cyl(a,b)$, of angle A, then any geodesic segment $[a,b]$ contains v, and $\mathrm{Ang}_v([a,b]) \geq A - 20D \geq A - \Theta$.

Proof: The second assertion is an immediate corollary of the first one, and of Definition 2.11.

Let w and w' be vertices in $Cyl(a,b)$, and v be such that $|w-v| + |v-w'| = |w-w'| \leq 50\delta$ in $\mathcal{K}$, and such that $\mathrm{Ang}_v([v,w],[v,w']) = A \geq \Theta$, for some geodesic segments $[v,w]$ and $[v,w']$.

As w is in the cylinder $Cyl(a,b)$, there exists a l-coarse-piecewise-geodesic $f\colon [0,T] \to \mathcal{K}$, with $f(0) = a$ and $f(T) = b$, such that $f(s) = w$ for some $s \in [0,T]$, and such that w is on a sub-local geodesic $f|_{[r,t]}$ of f, $|r-s|$ (resp. $|s-t|$) being larger than 10μ, except if $r = 0$ (resp. $(t = T)$.

As f is a quasi-geodesic, at least one of the segments $f|_{[s,t]}$ and $f|_{[r,s]}$ does not contain v. Let us assume that $f|_{[r,s]}$ does not contain v. We set $s_1 = \max\{0, s-3D\}$, and we choose x in a geodesic segment $[a,b]$ such that the distance $|x - f(s_1)|$ is minimal (it is less than D, and it is equal to 0 if $s_1 = 0$). Let $[x, f(s_1)]$ be a geodesic segment. We claim that this segment does not contain v. If $s_1 = 0$, the segment is exactly one point equal to a, and it cannot be v since a is never a parabolic slice. If $s_1 = s - 3D$, the subpath $[f(s_1), w]$ of f is included in a μ-local geodesic, and is of length $3D < \mu$. Hence it is a geodesic, and therefore $|f(s_1) - w| = 3D$. By triangular inequality, $|f(s_1) - v| \geq 3D - 50\delta > |f(s_1) - x|$, and therefore $[f(s_1), x]$ does not contain v, which is the claim.

Therefore, there is a path p from w to x of length at most $4D$ not containing v.

We do the same construction for w': there exist x' on $[a,b]$ and a path p' from w' to x' of length at most $4D$, not containing v. By triangular inequality, $|x-x'| \leq 8D + 50\delta \leq 9D$.

We now consider the path obtained by concatenation of p, $[x,x']$ and p' (with reverse orientation). Its length is at most $17D < A - 50\delta$. Therefore, the segment $[x,x']$ must contain v, and the triangular inequality for angles (Proposition 1.4 (1)) shows that $\mathrm{Ang}_v([x,x']) \geq A - 17D$. ∎

LEMMA 2.13 (Angles at the end of cylinders): *Let $x \neq b$ be in $Cyl(a,b)$. Then for all geodesic segments $[a,b]$ and $[x,b]$, $\mathrm{Ang}_b([x,b],[a,b]) \leq 14D$.*

Proof: We distinguish two cases. First assume that $|x-b| \geq 3D$. We know that there is a vertex w on the segment $[a,b]$ such that $|w-x| \leq D$. Therefore, in a geodesic triangle (b,w,x), the segments $[b,x]$ and $[b,w]$ remain δ-close for a length at least $D \geq 10\delta$. Therefore, their angle at b is less than 21δ, and it is less than $14D$.

Secondly, assume that $|x-b| \leq 3D$. There is a coarse-piecewise-geodesics $f\colon [0,T] \to \mathcal{K}$ between a and b, containing x on one of its sub-local geodesics. Let t be such that $f(t) = x$. Consider $t_1 = max\{0, t-3D\}$, and we choose $w \in [a,b]$ such that the distance $|w - f(t_1)|$ is minimal (it is less than D in any case, and it is 0 if $t_1 = 0$). Now we consider the path p obtained by the concatenation of a geodesic segment $[w, f(t_1)]$ (of length at most D), of $f|_{[t_1,t]}$ (of length at most $3D$), of a geodesic segment $[x,b]$ (of length at most $3D$), and of a subsegment $[b,w] \subset [b,a]$ (of length at most $7D$ by triangular inequality). As f is a quasi-geodesic, and $f(T) = b$, we deduce that b is not on the path $f|_{[t_1,t]}$. It is not on the segment $[w, f(t_1)]$ because $|w - f(t_1)| \leq |f(t_1) - b|$. Therefore, the path p passes only once at the vertex b, and therefore $\mathrm{Ang}_b([x,b],[b,a]) \leq 14D$.

We see that in any case, $\mathrm{Ang}_b([x,b],[b,a]) \leq 14D$. ∎

LEMMA 2.14 (Angles in a cylinder): *Let $[a,b]$ be a geodesic segment such that for some vertex v in $[a,b]$, one has $\mathrm{Ang}_v([a,b]) > \Theta - 20D$. Then, $Cyl(a,b) = Cyl(a,v) \cup Cyl(v,b)$.*

In particular, if $\{v\}$ is a parabolic slice of $Cyl(a,b)$, then

$$Cyl(a,b) = Cyl(a,v) \cup Cyl(v,b).$$

Moreover, in such a case, $Cyl(a,v) \cap Cyl(v,b) = \{v\}$.

Proof: Let w be a point of $Cyl(a,b)$. There exists $f\colon [0,T] \to \mathcal{K}$, a l-coarse-piecewise-geodesic from a to b that contains $w = f(s)$ on one of its sub-local-geodesic, with the condition of Definition 2.5. This coarse-piecewise-geodesic is a λ-quasi-geodesic, hence stays D-close to the segment $[a,b]$. Hence, by an argument similar to Lemma 1.5, any such coarse-piecewise-geodesic passes at the vertex v. Let t be the real number such that $f(t) = v$. Then, by Remark 2, with the induced subdivision, $f|_{[0,t]}$ is a l-coarse-piecewise-geodesic from a to v, and $f|_{[t,T]}$ is a l-coarse-piecewise-geodesic from v to b. Therefore, if $s \leq t$, we have that $w \in Cyl(a,v)$, and if $s \geq t$, then $w \in Cyl(v,b)$. This proves that $Cyl(a,b) \subset Cyl(a,v) \cup Cyl(v,b)$.

Let us prove the other inclusion. Let w be a point of $Cyl(a,v)$. There exists a l-coarse-piecewise-geodesic $f\colon [0,T] \to \mathcal{K}$ from a to v containing w on one of its sub-local-geodesics, with the condition of Definition 2.5.

Let $T' = T + |v-b|$, and let $\tilde{f}\colon [0,T'] \to \mathcal{K}$ be as follows: $\tilde{f}|_{[0,T]} \equiv f$, and $\tilde{f}(T+t)$ is the point of the given geodesic $[a,b]$ at distance $T'-T-t$ from b. Let $f|_{[c,T]}$ be the last sub-local geodesic of f, hence ending at v. Then $\tilde{f}|_{[c,T']}$ is still a μ-local-geodesic, by Lemma 1.5. Moreover, any subpath of length $1000\delta\frac{\lambda}{2} \leq \mu$ is a $\lambda/2$-quasi-geodesic: either it is included in the path f, or in the geodesic segments $[v,b]$, or it is the union of two geodesic segments that meet at v with an angle greater than $\Theta - 20D$, and therefore is geodesic by Lemma 1.5. Finally, $\tilde{f}$ stays at distance ϵ from a geodesic segment $[a,b]$. Therefore, $\tilde{f}$ is a l-coarse-piecewise-geodesic from a to b, coinciding with f between a and v. This proves that the point w is in $Cyl(a,b)$, and therefore $Cyl(a,v) \subset Cyl(a,b)$.

Similarly, by changing the role of a and b, one has $Cyl(v,b) \subset Cyl(a,b)$ and therefore $Cyl(a,b) = Cyl(a,v) \cup Cyl(v,b)$.

The second assertion of the lemma is a consequence of Lemma 2.12.

Let us prove now that the intersection $Cyl(a,v) \cap Cyl(v,b)$ is $\{v\}$. Let x be in the intersection $Cyl(a,v) \cap Cyl(v,b)$, and assume that $x \neq v$. By Lemma 2.13, $\mathrm{Ang}_v([x,v],[v,a]) \leq 14D$. Similarly, as x is also in $Cyl(v,b)$, $\mathrm{Ang}_v([x,v],[v,b]) \leq 14D$. The triangular inequality for angles (Proposition 1.4 (1)) indicates that $\mathrm{Ang}_v([a,v],[v,b])$ is at most $28D$, and contradicts the assumption that it is greater than $\Theta - 20D$. This proves that $Cyl(a,v) \cap Cyl(v,b) = \{v\}$. ∎

The lemma we have just proved allows us to consider unions of cylinders without parabolic slice. This enables the construction of regular slices, as in Rips and Sela [RS].

Let $Cyl(a,b)$ be a cylinder *without parabolic slice*, and $x \in Cyl(a,b)$. Following [RS], we define the set $N_R^{(a,b)}(x)$ as follows: it is the set of all the vertices $v \in Cyl(a,b)$ such that $|a-x| < |a-v|$, and such that $|x-v| > 100\delta$. Here R stands for "right", and $N_L^{(a,b)}(x)$ is similarly defined changing the condition $|a-x| < |a-v|$ into $|a-x| > |a-v|$. As cylinders are finite, those sets are also finite.

Definition 2.15 (Difference in cylinders without parabolic slice)[RS] 3.3*:* Let $Cyl(a,b)$ be a cylinder without parabolic slice, and x, y two points in it. We define

$$\begin{aligned}\mathrm{Diff}_{a,b}(x,y) =& \mathrm{Card}(N_L^{(a,b)}(x) \setminus N_L^{(a,b)}(y)) - \mathrm{Card}(N_L^{(a,b)}(y) \setminus N_L^{(a,b)}(x)) \\ &+ \mathrm{Card}(N_R^{(a,b)}(y) \setminus N_R^{(a,b)}(x)) - \mathrm{Card}(N_R^{(a,b)}(x) \setminus N_R^{(a,b)}(y)),\end{aligned}$$

where $\mathrm{Card}(X)$ is the cardinality of the set X.

Let us remark that this defines a cocycle (see [RS]).

Definition 2.16 (Regular slices in a cylinder without parabolic slice)*:* Let $Cyl(a,b)$ be a cylinder without parabolic slice. An equivalence class in $(Cyl(a,b) \setminus \{a,b\})$ for the equivalence relation $(\mathrm{Diff}_{a,b}(x,y) = 0)$ is called a **regular slice** of $Cyl(a,b)$.

Ordering of slices. We assign an index to each slice of $Cyl(a,b)$ as follows. Let $v_1, \ldots, v_k$ be the consecutive parabolic slices, ordered by their position on a geodesic segment $[a,b]$. We set S_0 to be $\{a\}$. For $j \geq 0$, we then define S_{j+1} to be the unique regular slice of the cylinder $Cyl(a,v_1)$ such that $\mathrm{Diff}(S_j, S_{j+1})$ is minimal. If S_j is the last slice in $Cyl(a,v_1)$, then the parabolic slice $\{v_1\}$ is labeled S_{j+1}. Then among the regular slices of a cylinder $Cyl(v_i, v_{i+1})$, we define S_{j+1} to be the (unique) slice such that $\mathrm{Diff}(S_j, S_{j+1})$ is minimal. If S_m is the last regular slice of a cylinder $Cyl(v_i, v_{i+1})$ (for $i < k$), then the parabolic slice $\{v_{i+1}\}$ is S_{m+1}. Finally, we order the slices of the last cylinder $Cyl(v_k, b)$ in the same way, and $\{b\}$ is the last slice (see Figure 3).

LEMMA 2.17: *Let a and b be two points in $\mathcal{K}$, and let v be in $Cyl(a,b)$. Let $[a,b]$ be a geodesic segment. Then there exists $w \in [a,b]$ such that $|w - v| \leq 2\delta$.*

Proof: The vertex v is on a sub-local geodesic of some coarse-piecewise-geodesic f. By definition of the elements of cylinders, there is a geodesic segment $[f(t_1), f(t_2)]$ containing v such that, for $i = 1,2$, $f(t_i)$ is at distance at most D of a point $w_i \in [a,b]$, and such that *either* $|v - f(t_i)| \geq 5D$, *or* $f(t_i)$ equals a or b (in this case, we choose w_i to be $f(t_i)$). The triangle $(w_1, f(t_1), f(t_2))$ is δ-thin. If the segment $[w_1, f(t_1)]$ is not reduced to a point, it remains at distance at least $4D$ from v, therefore v is at distance at most δ from a point v' of $[w_1, f(t_2)]$. Similarly, the triangle $(w_1, w_2, f(t_2))$ is δ-thin, and therefore v' is at distance at most δ from $[w_1, w_2]$, and v is at distance at most 2δ from the segment $[w_1, w_2] \subset [a,b]$. ∎

LEMMA 2.18: *Let $Cyl(a,b)$ be a cylinder, and let x and y be two points in $Cyl(a,b)$. Assume that there is a vertex v in some geodesic segment $[x,y]$ such* *that* $\mathrm{Ang}_v([x,y]) \geq 2\Theta$. *Then,* $\{v\}$ *is a parabolic slice of* $Cyl(a,b)$, *and if* $x \in Cyl(a,v)$ *then* $y \in Cyl(v,b)$.

Proof: If $|x-y| \leq 50\delta$, the result is a consequence of Lemma 2.12.

If $|x-y| \geq 50\delta$, let us parametrize the segment $[x,y]$ by arc length: $g\colon [0,L] \to \mathcal{K}$, and let $g(t) = v$. By the previous lemma, x and y are 2δ-close to a geodesic segment $[a,b]$. Let w and w' be points in $[a,b]$ realizing this distance. By hyperbolicity, the triangles (x,y,w) and (x,w,w') are δ-thin. Therefore, $g(t-5\delta)$ and $g(t+5\delta)$ are 2δ-close to the segment $[w,w'] \subset [a,b]$. Let us consider the path obtained by concatenation of geodesic segment: $[g(t+5\delta),y][y,w'][w',w][w,x][x,g(t-5\delta)]$, where the first and the last segments are included in $[x,y]$, and where $[w',w] \subset [b,a]$. By triangular inequality, its total length is at most 18δ. We assumed that $\mathrm{Ang}_v([x,y]) \geq 2\Theta$, therefore, it must contain v, and the only possibility is that $v \in [w',w] \subset [b,a]$; moreover, $\mathrm{Ang}_v([w',w]) \geq 2\Theta - 28\delta$. By the third assertion of Lemma 2.6, every vertex of the segment $[a,b]$ is in the cylinder $Cyl([a,b])$. Therefore, $\{v\}$ is a parabolic slice of $Cyl(a,b)$.

The second statement is a corollary of Lemma 2.13. ∎

LEMMA 2.19 (Slices are small): *Let* a *and* b *be two points of* $\mathcal{K}$, *and let* S *be a slice of* $Cyl(a,b)$. *If* v *and* v' *are in* S, *then* $|v-v'| \leq 200\delta$, *and for all geodesic segment* $[v,v']$ *one has* $\mathrm{MaxAng}([v,v']) \leq 2\Theta$.

Proof: If the slice S is parabolic, then $v = v'$, and there is nothing to prove. We can assume that the slice S is regular. We assume without loss of generality that $|a-v| \leq |a-v'|$.

Suppose that $|v-v'| \geq 200\delta$. Let us prove that $N_L^{(a,b)}(v) \subset N_L^{(a,b)}(v')$.

By Lemma 2.17, there is a vertex w on a geodesic segment $[a,b]$ such that $|w-v| \leq 2\delta$, and similarly, there is w' on $[a,b]$ such that $|w'-v'| \leq 2\delta$. Note that, since $|v-v'| \geq 200\delta$, the distance $|w'-w|$ is at least 196δ. The points w and w' are both on the segment $[a,b]$, therefore, if $|a-w| > |a-w'|$, then $|a-w| \geq |a-w'| + 196\delta$, and $|a-v| > |a-v'|$, contradicting our assumption. Therefore, $|a-w| = |a-w| - |w-w'| \leq |a-w'| - 196\delta$.

Let z be in $N_L^{(a,b)}(v)$. As it is an element of the cylinder $Cyl(a,b)$, by Lemma 2.17, there is a vertex w_z of $[a,b]$ such that $|z-w_z| \leq 2\delta$. By definition, the vertex z is at distance at least 100δ from v, therefore $|w-w_z| \geq 96\delta$.

Moreover, as $|a-z| \leq |a-v|$ and $|z-v| \geq 100\delta$, the vertex w_z is on the subsegment $[a,w]$ of $[a,b]$. Therefore, $|w_z-w'| = |w_z-w| + |w-w'| \geq 292\delta$. This gives, by triangular inequality, $|z-v'| \geq 290\delta$. Therefore, z is in $N_L^{(a,b)}(v')$.

Hence, we have $N_L^{(a,b)}(v) \subset N_L^{(a,b)}(v')$ and similarly $N_R^{(a,b)}(v') \subset N_R^{(a,b)}(v)$. Moreover, $N_L^{(a,b)}(v) \neq N_L^{(a,b)}(v')$ (and similarly $N_R^{(a,b)}(v') \neq N_R^{(a,b)}(v)$), because v' is in $N_L^{(a,b)}(v)$ and not in $N_L^{(a,b)}(v')$. Therefore, $\mathrm{Diff}_{a,b}(v,v') \neq 0$, which is a contradiction since they both are in the same regular slice.

The bound on the maximal angle of a geodesic segment $[v,v']$ is a corollary of Lemma 2.18: if $\mathrm{Ang}_w([v,v']) \geq 2\Theta$ for some w, Lemma 2.18 implies that v and w are not in the same slice (not even in consecutive slices). ∎

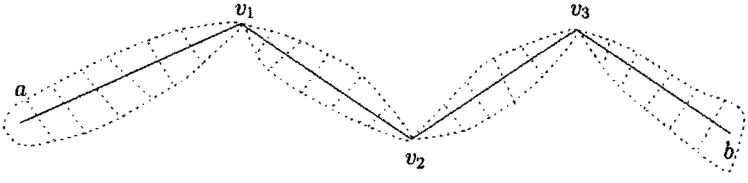


Figure 3. Regular and parabolic slices in a cylinder.

Corollary 2.20 (Consecutive slices are close): *Let $Cyl(a,b)$ be a cylinder, and let S and S' be two consecutive slices. Let $v \in S$ and $v' \in S'$.*

Then $|v - v'| \leq 1000\delta$ and $\mathrm{MaxAng}([v,v']) \leq 2\Theta$.

Proof: The bound of the maximal angle is a consequence of Lemma 2.18: if there was such an angle there would be a parabolic slice between S and S', hence they would not be consecutive.

Assume that $|v-v'| \geq 1000\delta$, and without loss of generality, $|a-v| \leq |a-v'|$. By Lemma 2.17, the points v and v' are 2δ-close to a geodesic segment $[a,b]$. Let w be on $[a,b]$, at distance at least 400δ from v and v', and such that $|a-v| \leq |a-w| - 200\delta \leq |a-w| + 200\delta \leq |a-v'|$. By Lemma 2.19, w is not in S nor in S', and as it is on a geodesic segment $[a,b]$, it is in a slice. This slice is not before S and not after S', therefore S and S' are not consecutive. ∎

Lemma 2.21 (Locality of the regular slices): *Let $Cyl(a,b)$ and $Cyl(a,c)$ be cylinders without parabolic slices, and let R be a number. Assume that $Cyl(a,b) \cap B_R(a) = Cyl(a,c) \cap B_R(a)$, where $B_R(a)$ is the ball centered at a of radius R. Then, any slice of $Cyl(a,b)$ included in $B_{R-200\delta}(a)$ is a slice of $Cyl(a,c)$.*

Proof: Let S be a slice of $Cyl(a,b)$ and assume that S is included in $B_{R-200\delta}(a)$. Let v be a point in S. There exists S', a slice of $Cyl(a,c)$ containing v. Let v' be an arbitrary element of S. We claim that v' is in S'.

Let us compute $\mathrm{Diff}_{a,c}(v,v')$. It is equal to

$$\begin{aligned}&\mathrm{Card}(N_L^{(a,c)}(v) \setminus N_L^{(a,c)}(v')) - \mathrm{Card}(N_L^{(a,c)}(v') \setminus N_L^{(a,c)}(v))\\ &+ \mathrm{Card}(N_R^{(a,c)}(v') \setminus N_R^{(a,c)}(v)) - \mathrm{Card}(N_R^{(a,c)}(v) \setminus N_R^{(a,c)}(v')).\end{aligned}$$

Note that $N_L^{(a,c)}(v) = N_L^{(a,b)}(v)$, and similarly for v'. If x is in $N_R^{(a,c)}(v') \setminus N_R^{(a,c)}(v)$, then it is 100δ-close to v. Therefore, x is in $Cyl(a,b)$, and it is in $N_R^{(a,b)}(v') \setminus N_R^{(a,b)}(v)$. Similarly, the other inclusion holds and one has $N_R^{(a,c)}(v') \setminus N_R^{(a,c)}(v) = N_R^{(a,b)}(v') \setminus N_R^{(a,b)}(v)$.

Therefore, $\mathrm{Diff}_{a,c}(v,v') = \mathrm{Diff}_{a,b}(v,v')$, and it is equal to 0 since we assumed that $v' \in S$. Therefore, $v' \in S'$, and we deduce that $S' \subset S$. Similarly, one has the other inclusion, and $S = S'$. This proves the lemma. ∎

THEOREM 2.22 (Coincidence of the decomposition in slices): *With the notation of Theorem 2.9, let $(x,y,z) = (p, \alpha p, \gamma^{-1}p)$ be a triangle in $\mathcal{K}$ such that α, β, γ are in $F \cup F^{-1}$, and $\alpha\beta\gamma = 1$.*

The ordered slice decomposition of the cylinders is as follows.

$$\begin{aligned}Cyl(x,y) &= (S_1, S_2, \ldots, S_k, \mathcal{H}_z, T_m, T_{m-1}, \ldots, T_1),\\ Cyl(x,z) &= (S_1, S_2, \ldots, S_k, \mathcal{H}_y, V_p, V_{p-1}, \ldots, V_1),\\ Cyl(y,z) &= (T_1, T_2, \ldots, T_m, \mathcal{H}_x, V_p, V_{p-1}, \ldots, V_1),\end{aligned}$$

where $S_1, \ldots, S_k, T_1, \ldots, T_m$ and $V_1, \ldots, V_p$ are slices and where each $\mathcal{H}_v$ ($v = x,y,z$) is a set of at most $10\varphi(n)$ consecutive slices, without parabolic slice of angle more than $3\Theta + 100\delta$.

The sets $\mathcal{H}_v$ are called the **holes** *of the slice decomposition.*

Proof: Consider the cylinders $Cyl(x,y)$ and $Cyl(x,z)$. By Theorem 2.9, they coincide in $B_{R_{x,y,z}}(x)$. Therefore, any parabolic slice of $Cyl(x,y)$ that is located in $B_{R_{x,y,z}-2}(x)$ is also a parabolic slice of $Cyl(x,z)$, and similarly, permuting x and y.

Let $\{v\}$ be the last common parabolic slice of these two cylinders, or $v = x$ if they have no common parabolic slice: $Cyl(x,y) = Cyl(x,v) \cup Cyl(v,y)$ and $Cyl(x,z) = Cyl(x,v) \cup Cyl(v,z)$, by Lemma 2.14.

The ordered slices of the cylinders $Cyl(x,y)$ and $Cyl(x,z)$ obviously coincide at least until the slice $\{v\}$.

Let $\{w\}$ be the first parabolic slice of $Cyl(x,y)$ after $\{v\}$, or $w = y$ if there is no such parabolic slice. Let $\{w'\}$ be the first parabolic slice of $Cyl(x,y)$ after $\{v\}$, or $w' = z$ if there is none. By Theorem 2.9, $Cyl(v,w) \cap B_{R_{x,y,z}-|x-v|}(v) = Cyl(v,w') \cap B_{R_{x,y,z}-|x-v|}(v)$. These cylinders are without parabolic slices. By Lemma 2.21, their regular slices lying in $B_{R_{x,y,z}-|x-v|-200\delta}(v)$ coincide.

In other words, the slice decomposition of $Cyl(x,y)$ and $Cyl(x,z)$ coincide at least until their last common parabolic slice, and for all slices in $B_{(R_{x,y,z}-200\delta)}(x)$. A similar statement holds for the other pairs of cylinders.

It remains to prove that no hole contains a parabolic slice of angle greater than $3\Theta + 100\delta$. Let us consider such a parabolic slice $S = \{v\}$ in $Cyl(x,y)$. By Lemma 2.12, a segment $[x,y]$ has an angle greater than $2\Theta + 100\delta$ at the point v. Therefore, one of the two segments $[x,z]$ and $[z,y]$ (say $[x,z]$) has an angle greater than Θ at v, and we deduce that $\{v\}$ is a parabolic slice of $Cyl(x,z)$. As it is simultaneously a parabolic slice in $Cyl(x,y)$ and in $Cyl(x,z)$, it is not in a hole. ∎

3. Image of a group in a relatively hyperbolic group

In this section we consider Γ a relatively hyperbolic group with associated graph $\mathcal{K}$, and G a finitely presented group with a morphism $h\colon G \to \Gamma$. We want to explain how to adapt Delzant's method, given for hyperbolic group in [Del], to the relative case, in order to obtain an analogue to Thurston's Theorem 0.1.

For convenience, we choose the graph $\mathcal{K}$ with the following four properties:

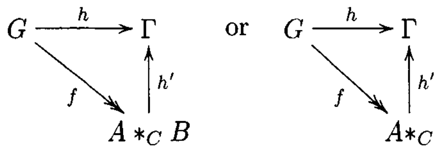
It has a base point p with trivial stabilizer. Its vertices are exactly the infinite valence vertices and the elements of the orbit of p. It has no pair of adjacent vertices of infinite valence. It is possible to choose $\mathcal{K}$ satisfying these requirements: see, for example, the coned-off graph of a Cayley graph in [F].

In $\mathcal{K}$, for all L, there are, by definition of fineness, only finitely many orbits of pairs of edges sharing an infinite valence vertex, that make an angle less than L. Given a word metric on Γ, let $\Psi(L)$ be a number bounding the distance in $Cay(\Gamma)$ between their non-common vertices. Then, one has for all γ in Γ, for every geodesic segment $[p, \gamma p]$ in $\mathcal{K}$,

$$|\gamma p - p| \times (\Psi(\mathrm{MaxAng}([p, \gamma p])) + 1) \geq |\gamma|. \tag{1}$$

Remark 3: In such a graph, a cylinder cannot have two consecutive parabolic slices: a geodesic segment between two parabolic slices $\{v_1\}$ and $\{v_2\}$ must contain a vertex with trivial stabilizer, which belongs to some regular slice of $Cyl(v_1, v_2)$.

Definition 3.1 (Accidental parabolic): We say that the morphism $h\colon G \to \Gamma$ has an accidental parabolic either if $h(G)$ is parabolic in Γ, or if there exists a non-trivial amalgamated free product $A *_C B$, or an HNN extension $A*_C$, and a factorization of h:



such that f is surjective and the image of C by h' is a finite or parabolic subgroup of Γ.

LEMMA 3.2: *If a subgroup H of Γ has a finite orbit in the graph $\mathcal{K}$, then either H is finite or it is parabolic.*

Proof: The subgroup H has a subgroup of finite index $\tilde{H}$, fixing a point in $\mathcal{K}$. Assume that H is infinite, and not equal to $\tilde{H}$. As $\tilde{H}$ is also infinite, it is parabolic, and the intersection of all its conjugates in H is infinite. But it is easily seen from fineness that the intersection of two distinct conjugates of a maximal parabolic subgroup is finite in a relatively hyperbolic group. Hence, H is itself parabolic. ∎

In the rest of this section, we prove the next theorem.

THEOREM 3.3: *Let G be a finitely presented group, and Γ a relatively hyperbolic group. There is a finite family of subgroups of Γ such that the image of G by any morphism $h\colon G \to \Gamma$ without accidental parabolic is conjugated to one of them.*

Proof: Let h be a morphism $h\colon G \to \Gamma$. We will construct a factorization of h through a certain graph of groups, and then we will deduce that either h has an accidental parabolic, or $h(G)$ is conjugated to a subgroup of Γ generated by small elements.

We choose a triangular presentation of G: $G = < g_1, \dots, g_k | T_1, \dots, T_n >$ with n relations which are words of three (or two) letters. This defines a Van Kampen polyhedron for G, which consists of one vertex, k edges, and n triangles and digons, and we denote P to be this polyhedron.

Recall that the base point p of the graph $\mathcal{K}$ associated to the relatively hyperbolic group Γ has trivial stabilizer. We consider the cylinders of the triangles, and their decomposition in slices obtained by Theorems 2.9 and 2.22, for the family $F = \{h(g_1), \dots, h(g_k)\} \subset \Gamma$ and the base point $p \in \mathcal{K}$.

3.1 THE LAMINATION Λ ON P. We now define a lamination on P, in two steps: first by choosing markings on the edges of P, and secondly by defining arcs in P between these markings.

3.1.1 Markings on the edges of P. For a generator g_i of G, let L_i^r be the number of regular slices of the cylinder of $[p, h(g_i)p]$ in $\mathcal{K}$, and L_i^p the number of its parabolic slices. Let c_i be the loop of the polyhedron P canonically associated to g_i. Let $m_i^1, \dots, m_i^{L_i^r+2L_i^p}$ be $(L_i^r + 2L_i^p)$ points on c_i, such that, if c_i: $[0,1] \to P$ is an arc-length parametrization of c_i, one has

$$m_i^k = c_i\Big(k\frac{1}{L_i^r + 2L_i^p + 1}\Big).$$

We call them the **markings** of the slice decomposition on c_i. To each marking of c_i we associate a slice in the cylinder of $[p, h(g_i)p]$ in $\mathcal{K}$: m_i^1 is associated to the first slice; if m_i^k is associated to a regular slice, m_i^{k+1} is associated to the next slice in the ordering, if m_i^k is associated to a parabolic slice, and if m_i^{k-1} is associated to another slice (or does not exist), then m_i^{k+1} is associated to the same slice as m_i^k; finally, if m_i^k and m_i^{k-1} are associated to the same parabolic slice, then m_i^{k+1} is associated to the next slice in the ordering. Note that every regular slice has one marking on c_i associated to it, and every parabolic slice has two markings.

3.1.2 Regular arcs in a triangle (or a digon) of P. The lamination Λ will be defined on P by its intersection with each triangle or digon T in P.

Consider a triangle T (with a euclidean metric) of P whose edges c_i, c_j, c_k correspond to the relation $g_i g_j g_k = 1$ of the presentation.

Consider two markings, m_i^r of c_i and m_j^s of c_j, that are associated to the same regular slice in the cylinders of the triangle $(p, h(g_i)p, h(g_k^{-1})p)$ in $\mathcal{K}$. The segment $[m_i^r, m_j^s]$ in T is said to be a **regular arc**.

Consider two consecutive markings, m_i^r and m_i^{r+1}, of c_i, associated to the same parabolic slice of $Cyl([p, h(g_i)p])$. There are three possibilities.

First, if the slice is not equal to a slice of any of the other two cylinders (that is, if it is in a hole in the sense of Theorem 2.22), we do nothing.

Secondly, if it is a slice of exactly one other cylinder, say, for example, $Cyl([h(g_i)p, h(g_j)h(g_i)p])$, then there are two consecutive markings m_j^s and m_j^{s+1} of c_j associated to it. The segments $[m_i^r, m_j^{s+1}]$ and $[m_i^{r+1}, m_j^s]$ are also said to be **regular arcs**. Note that these two segments do not cross.

Finally, if the slice is also a slice of the other two cylinders $Cyl([h(g_k^{-1})p, p])$ and $Cyl([h(g_i)p, h(g_k^{-1})p])$, there are two consecutive markings m_j^s and m_j^{s+1} of c_j, and two consecutive markings m_k^t and m_k^{t+1} of c_k, associated to it. The three segments $[m_i^r, m_k^{t+1}]$, $[m_i^{r+1}, m_j^s]$ and $[m_j^{s+1}, m_k^t]$ are **regular arcs**. These three segments do not cross each other.

We proceed similarly after cyclic permutations of i, j and k. We denote by $\Lambda_r(T)$ the union of all the regular arcs in T.

3.1.3 Singular arcs in a triangle (or a digon) of P. If the slice decomposition of the triangle has a hole (in the sense of Theorem 2.22), there are markings that are not in regular arcs. In such a case, we add a **singular point** p_T in the component of $T \setminus \Lambda_r(T)$ containing these markings. For every marking m not in $\Lambda_r(T)$, the segment $[m, p_T]$ is said to be a **singular arc**. Let $\Lambda_s(T)$ be the union of these singular arcs in T.

The lamination Λ on P is defined by: for every triangle or digon T of P, $\Lambda \cap T = \Lambda_r(T) \cup \Lambda_s(T)$ (see Figure 4).

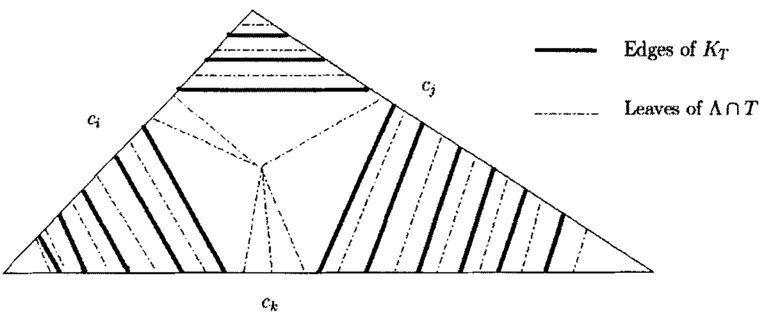


Figure 4. The lamination $\Lambda \cap T$ and the graph K_T in a triangle T of P.

3.2 Graph K on P. In each triangle or digon T of P, we draw a (disconnected) graph K_T satisfying: each connected component of $T \setminus K_T$ contains one and only one leaf of $\Lambda \cap T$, and its intersection $K_T \cap c_i$ with the edges c_i of T consists of the vertices of K_T; moreover, they are located on middles of consecutive markings of g_i (see Figure 4).

Let K be the union of all those graphs: $K = \bigcup_{i=1}^n K_{T_i}$. Some of the components of K have edges with one vertex in a hole of a slice decomposition. Let K' be the graph obtained from K when one has removed all these components.

There are two kinds of connected components of K': the components K_i for which a small tubular neighborhood NK_i is such that $NK_i \setminus K_i$ is disconnected (type I), and those for which it is connected (type II).

3.3 G AS A GRAPH OF GROUPS. We now split G in a graph of groups by cutting P along the graph K.

The graph of groups we consider is as follows. Its vertices are of two kinds. First, there are the connected components of $P \setminus K'$, and the groups are the fundamental groups of those components. There are also the components of K' of type II, and the groups are the fundamental groups of a small tubular neighborhood. The edges of the graph of groups are the components K_i of K', and their groups are either $\pi_1(NK_i)$, the fundamental group of a small tubular neighborhood, in the case of a component of type I, or $\pi_1(NK_i \setminus K_i)$ otherwise, in type II. Note that in this case, $\pi_1(NK_i \setminus K_i)$ is of index two in $\pi_1(NK_i)$.

LEMMA 3.4 ([Del] Lemma III.2): *Let H be a subgroup of G conjugated to an edge group of the graph of groups, or a vertex not corresponding to the component of a singular leaf of Λ. Then $h(H)$ is a subgroup of Γ that has an orbit in $\mathcal{K}$ which is contained in a slice. In particular, this orbit is finite.*

For the proof, see [Del].

In the case of hyperbolic groups, one deduces that the subgroup is finite; in our case, by Lemma 3.2, it is either finite or parabolic.

COROLLARY 3.5: *If the map h has no accidental parabolic, then the image by h of the graph of groups defined above is a trivial splitting of $h(G)$. In this case, $h(G)$ is the image of a vertex group corresponding to a leaf λ in P, containing singular points of the lamination: $h(G)$ is conjugated to the image of $\pi_1(\lambda)$ (only defined up to conjugacy).*

Proof: By assumption, the image of G by h is neither finite nor parabolic. If the graph of groups defined above induces a non-trivial splitting of $h(G)$, then, by the previous Lemma, every edge group of this splitting is finite or parabolic. Therefore, this splitting of $h(G)$ gives an accidental parabolic for h. This proves the first assertion: the image of the graph of groups is a trivial splitting of $h(G)$. As a consequence, $h(G)$ is equal to the image of a vertex group. As it is assumed to be neither parabolic nor finite, it has to be a vertex group corresponding to a singular leaf. ∎

3.4 IMAGES OF VERTEX GROUPS. The aim of the following is to study the image of singular leaves of Λ.

Let P_Γ be a Van Kampen polyhedron for Γ, for a presentation with a finite generating set: it is a cell complex of dimension 2, whose 1-skeleton consists of finitely many loops. The set of vertices of its universal cover is $\widetilde{P_\Gamma}^0$, and after the choice of a base point we identify it with Γ, hence also with $\Gamma p = \{\gamma p, \gamma \in \Gamma\} \subset \mathcal{K}$, the set of vertices of finite valence of $\mathcal{K}$: $\widetilde{P_\Gamma}^0 \approx \Gamma \approx (\Gamma p)$.

3.4.1 Lifting slices of $\mathcal{K}$ in Γ. From the identification above, we have an inclusion of $\widetilde{P_\Gamma}^0$ in $\mathcal{K}$. We want to define good pre-images in $\widetilde{P_\Gamma}^0 \approx \Gamma$ of slices of cylinders in $\mathcal{K}$. We define the pre-image S_Γ of an arbitrary slice S as follows.

If S is a regular slice of a cylinder in $\mathcal{K}$, which is *not* reduced to a vertex of infinite valence, then we say that S_Γ is

$$S_\Gamma = \{\gamma \in \Gamma | \exists s_1, s_2 \in S, |s_1 - \gamma p| + |\gamma p - s_2| = |s_1 - s_2|\},$$

the set of elements of Γ that send the base point p of $\mathcal{K}$ on a geodesic segment with ends in S.

If $S = \{v\}$ is a parabolic slice of a cylinder in $\mathcal{K}$, or a regular slice reduced to a point of infinite valence, then we define S_Γ to be the set $\{\gamma \in \Gamma || \gamma p - v| = 1\}$. It is a coset of a parabolic subgroup of Γ. Note that, in all cases, S_Γ is not empty.

3.4.2 The map $\tilde{h}$: $\tilde{P} \to \widetilde{P_\Gamma}$. Let $\tilde{P}$ be the universal cover of the polyhedron P, and $*$ a base point in it. For every $i = 1, \dots, k$, for every edge c_i in the 1-skeleton of P, we denote by $\tilde{c}_i$ its image in $\tilde{P}$ starting at $*$. We define markings on $\tilde{c}_i$ such that they map exactly on the ones of c_i by the quotient map, and we extend the construction by G-equivariance in $\tilde{P}$. Every edge of the 1-skeleton of $\tilde{P}$ is hence marked by consecutive markings.

Recall that $\widetilde{P_\Gamma}$ is the universal cover of P_Γ. In this part, we construct a suitable continuous G-equivariant map $\tilde{h}$: $\tilde{P} \to \widetilde{P_\Gamma}$ such that for all $i = 1, \dots, k$, $\tilde{h}(\tilde{c}_i)$ is a path from $\tilde{h}(*)$ to $h(g_i)\tilde{h}(*)$, where g_i denotes the element of G associated to c_i.

The map $\tilde{h}$ is completely defined on the vertices of $\tilde{P}$ by the equivariance formula: $\tilde{h}(gv) = h(g)\tilde{h}(v)$, for all $g \in G$. We now choose the images of the markings of each edge $\tilde{c}_i$ $(i = 1, \dots, k)$.

First, if m_i^j is any marking of c_i associated to a slice S (without restriction), and if $\tilde{m}_i^j$ is its image in $\tilde{c}_i$, then $\tilde{h}(\tilde{m}_i^j)$ is equal to a vertex $\gamma\tilde{h}(*)$ of $\widetilde{P_\Gamma}$ such that $\gamma \in S_\Gamma$.

Second, if m_i^j is a marking of c_i associated to a parabolic slice $S = \{v\}$, there is a unique marking adjacent to m_i^j in c_i which is associated to a slice $S' \neq S$. Then we require that $\tilde{h}(\tilde{m}_i^j) = \gamma\tilde{h}(*)$, where $\gamma \in S_\Gamma$ is such that γp lies on some geodesic from v to a point of S' in $Cyl(p, h(g_i)p)$. We denote by $S_\Gamma(i,j)$ the set of such elements $\gamma \in S_\Gamma$. Note that the images of the two markings of a parabolic slice might be very far from each other in $\widetilde{P_\Gamma}$, in the same coset of a parabolic subgroup.

Third, if m_i^j is a marking of c_i associated to a regular slice S reduced to a vertex of infinite valence $S = \{v\}$, then we require that $\tilde{h}(\tilde{m}_i^j) = \gamma\tilde{h}(*)$, where $\gamma \in S_\Gamma$ is such that γp lies on some geodesic from v to a point of a slice adjacent to S in $Cyl(p, h(g_i)p)$. We denote by $S_\Gamma(i,j)$ the set of such elements $\gamma \in S_\Gamma$.

We can assume that $\tilde{h}(\tilde{c}_i)$ is a geodesic between the images of consecutive markings, but this is not essential. By equivariance, $\tilde{h}$ is then defined on the 1-skeleton of $\tilde{P}$, and is continuous and equivariant. We extend $\tilde{h}$ on the triangles of $\tilde{P}$ using the simple connectivity of $\widetilde{P_\Gamma}$, so that the map is still continuous and equivariant. Hence $\tilde{h}\colon \tilde{P} \to \widetilde{P_\Gamma}$ is well defined.

Lemma 3.6: *Let v be a vertex of $\mathcal{K}$ of infinite valence such that $\{v\}$ is a slice (either parabolic or regular) of the cylinder $Cyl(p, h(g_i)p)$. Let m_i^j be a marking on c_i, associated to the slice $S = \{v\}$. The diameter of $S_\Gamma(i,j)$ in Γ (for the word metric) is at most $2000\delta(\Psi(2\Theta)+1)$.*

Proof: Let γ_1 and γ_2 be in $S_\Gamma(i,j)$. There are points $v_1 = \gamma_1 p$ and $v_2 = \gamma_2 p$ in slices S'_1 and S'_2 adjacent to S in $Cyl(p, h(g_i)p)$. By Corollary 2.20, $|v - v_i| \leq 1000\delta$, and for some geodesic segments, $\mathrm{MaxAng}([v, v_i]) \leq 2\Theta$, for $i = 1, 2$.

First, assume that S is a parabolic slice. Then, by the definition of $S_\Gamma(i,j)$, $S'_1 = S'_2$. By Lemma 2.13, $\mathrm{Ang}_v([v, v_1], [v, v_2]) \leq 14D \leq \Theta$. Therefore, by the majoration (1), we can deduce that $|\gamma_1^{-1}\gamma_2| \leq 2000\delta(\Psi(2\Theta)+1)$.

Secondly, assume that S is a regular slice. Then there is no parabolic slice between S'_1 and S'_2. By Lemma 2.18, $\mathrm{Ang}_v([v, v_1], [v, v_2]) \leq 2\Theta$. Therefore, again by the majoration (1), $|\gamma_1^{-1}\gamma_2| \leq 2000\delta(\Psi(2\Theta)+1)$. ∎

3.4.3 Bounding the lengths of the images of leaves of Λ in P_Γ. The equivariant and continuous map $\tilde{h}$ induces in the quotient a continuous map, which we denote $h\colon P \to P_\Gamma$, since it realizes the morphism of groups $h\colon \pi_1(P, *) \to \pi_1(P_\Gamma, *)$, where $*$ denotes the only vertex of the complexes.

The next lemma is an analogue of Lemma II.1 in [Del], but cannot be deduced from it because of the presence of parabolic slices. Recall that n is the number of relators of the given presentation of G, or the number of triangles and digons of P.

LEMMA 3.7: *Let $l_1, \dots, l_m$ be a sequence of regular arcs of Λ, where l_i links the marking $\iota(l_i)$ to the marking $\tau(l_i)$, and where $\tau(l_i) = \iota(l_{i+1})$. If the path $l_1 l_2 \cdots l_m$ has no loop, then the path $h(l_1 l_2 \cdots l_m)$ in P_Γ is homotopic, with fixed ends, to a path in the 1-skeleton of P_Γ, of length less than $20000\delta(\Psi(5\Theta)+1)n$ (for the graph metric of the 1-skeleton).*

Proof: As the arcs are all regular, all the markings involved are associated to the same slice of $\mathcal{K}$, say S. Let us lift the path $l_1 l_2 \cdots l_k$ in a path $\widetilde{l_1 l_2 \cdots l_k}$ of $\tilde{P}$, starting at the marking $\tilde{m}_i^j$, where $m_i^j = \iota(l_1)$. Thus, this path is mapped in $\widetilde{P_\Gamma}$ on a path that stays in S_Γ. As $\widetilde{P_\Gamma}$ is simply connected, this path is homotopic to any path in the 1-skeleton that has the same ends.

There are two main cases to study, namely if the slice is regular, not reduced to a single point of infinite valence, or if it is reduced to a single point of infinite valence (including the case of parabolic slices). In the second case, we will have to discuss whether an adjacent arc of the lamination is regular or not.

First, if the slice S is regular, not reduced to a single vertex of infinite valence, then the end points v_0 and v_m of $\tilde{h}(\widetilde{l_1 l_2 \cdots l_m})$ are vertices of the form $v_0 = \gamma_0 \tilde{h}(*)$ for $\gamma_0 \in S_\Gamma$, and $v_m = \gamma_m \tilde{h}(*)$ for $\gamma_m \in S_\Gamma$. Therefore, there exist s_0 and s_0' in S and a geodesic segment $[s_0, s_0']$ in $\mathcal{K}$ containing $\gamma_0 p$ (and similarly for γ_m). By Lemma 2.19, we have a path from $\gamma_0 p$ to $\gamma_m p$ of length at most $3\times 200\delta$, and of maximal angle at most 2Θ. Therefore, by the majoration (1), the distance in the 1-skeleton of $\widetilde{P_\Gamma}$ between v_0 and v_m is at most $600\delta(\Psi(2\Theta)+1)$.

Secondly, and in the rest of the proof, we assume that S is a parabolic slice or a regular slice reduced to a single vertex of infinite valence (in such a case, S_Γ is not bounded). We denote $S = \{v\} \subset \mathcal{K}$, and Δ_i the triangle or digon of $\tilde{P}$ containing $\tilde{l}_i$. We denote $C_{i,1}, C_{i,2}, C_{i,3}$ the three cylinders in $\mathcal{K}$ corresponding to the image of the sides of Δ_i in $\widetilde{P_\Gamma}$.

Then in the edge of Δ_i containing the marking $\widetilde{\iota(l_i)}$, there is one (and only one, if the slice is parabolic) marking $\widetilde{m_{\iota,i}}$ adjacent to $\widetilde{\iota(l_i)}$ that is not associated to S. In the edge containing the marking $\widetilde{\tau(l_i)}$, there is only one marking $\widetilde{m_{\tau,i}}$ adjacent to $\widetilde{\tau(l_i)}$ that is not associated to S, and that is linked to $\widetilde{m_{\iota,i}}$ by an arc (regular or singular) of the lamination of the triangle or digon. These markings are associated to regular slices (cf. Remark 3).

There are two possibilities.

In Δ_i, it is possible that $[\widetilde{m_{\iota,i}}, \widetilde{m_{\tau,i}}]$ be is a regular arc of $\tilde{\Lambda}$ (its projection in

Λ is a regular arc). Let $\widetilde{l_{i_0}\cdots l_{i_1}}$ be a maximal subpath such that this property holds at each step. By Lemma 3.6, the end points of the image by $\tilde{h}$ of $\widetilde{l_{i_0}\cdots l_{i_1}}$ in $\widetilde{P_\Gamma}$ are at distance at most $2000\delta(\Psi(2\Theta)+1)$ in the 1-skeleton of $\widetilde{P_\Gamma}$. Therefore, the image of $\widetilde{l_{i_0}\cdots l_{i_1}}$ in $\widetilde{P_\Gamma}$ is homotopic with fixed ends, to a path in the 1-skeleton of length less than $2000\delta(\Psi(2\Theta)+1)$.

Assume now that $[\widetilde{m_{\iota,i}}, \widetilde{m_{\tau,i}}]$ is not a regular arc of $\tilde{\Lambda}$, that is that $\tilde{l_i}$ is one of the three regular leaves of Δ_i which is adjacent to a singular leaf. Note that in a path $l_1\cdots l_m$ without loop, this can only happen $3n$ times, where n is the total number of triangles and digons.

Recall that $S=\{v\}$ is the slice in $\mathcal{K}$ of the cylinders $C_{i,1}$ and $C_{i,2}$ associated to $\widetilde{\iota(l_i)}$ and $\widetilde{\tau(l_i)}$. Let S_ι be the slice of $C_{i,1}$ associated to $\widetilde{m_{\iota,i}}$, and S_τ be the slice of $C_{i,2}$ associated to $\widetilde{m_{\tau,i}}$.

In order to bound the distance between the images by $\tilde{h}$ of $\widetilde{\iota(l_i)}$ and $\widetilde{\tau(l_i)}$ (both in S_Γ), it is enough to bound the length and the maximal angle of a geodesic between an element of S_ι and one of S_τ (in $\mathcal{K}$). Let v_ι be in S_ι, and v_τ in S_τ. By Corollary 2.20 they are at distance at most 2000δ in $\mathcal{K}$

We claim that, given a geodesic segment $[v_\iota, v_\tau]$ in $\mathcal{K}$, its maximal angle is at most 5Θ.

If $S=\{v\}$ is a regular slice, it is the triangular inequality for angles (Proposition 1.4 (1)) in the triangle (v_ι, v_τ, v).

If S is parabolic, we consider two geodesic segments $[v_\iota, v]$ and $[v, v_\tau]$ of $\mathcal{K}$. By Corollary 2.20, their maximal angle is at most 2Θ. If their angle is larger than 5Θ at v, by Lemma 1.5, their concatenation is a geodesic, and the geodesic of cylinder $C_{i,3}$ must contain v and make an angle greater than $5\Theta-3000\delta>\Theta$ there. Therefore, $\{v\}$ would be a parabolic slice of $C_{i,3}$, and by the construction of the leaves in a triangle the marking $\widetilde{\tau(l_i)}$ should be on the corresponding side of Δ_i, which is not the case. This proves our claim.

Thus, given a geodesic segment $[v_\iota, v_\tau]$ in $\mathcal{K}$, its maximal angle is at most 5Θ, and its length is at most 2000δ. Therefore, the distance between the images of $\widetilde{\iota(l_i)}$ and $\widetilde{\tau(l_i)}$ in the 1-skeleton of $\widetilde{P_\Gamma}$ is at most $3000\delta(\Psi(5\Theta)+1)$.

For a path $l_1 l_2\cdots l_m$ without loop, such a situation can happen only $3n$ times, where n is the number of triangles. Therefore, the distance between the endpoints of its image, in the 1-skeleton of $\widetilde{P_\Gamma}$, is at most

$$3n\times(2000\delta(\Psi(2\Theta)+1)+3000\delta(\Psi(5\Theta)+1))+2000\delta(\Psi(2\Theta)+1).$$

This is less than $20000\delta(\Psi(5\Theta)+1)\times n$. ∎

LEMMA 3.8: *An arc of Λ linking two markings corresponding to slices in the hole of a same triangle (as defined in Theorem 2.22), maps in P_Γ by h, on a path which is homotopic, with fixed ends, to a path in the 1-skeleton of P_Γ, of length less than* $(\varphi(n)+1)\times(20000\delta(\Psi(5\Theta)+1))$.

Proof: Such an arc is homotopic with fixed ends in P to a path tracking back on the side of the triangle containing the first marking, until the first regular arc to the other side, and then tracking on this side to the second marking. The image of this path by h is homotopic with fixed ends to the image of the arc of Λ. By Theorem 2.22, this path enters at most $2\times(10\varphi(n)+1)$ slices, none of them having an angle greater than 5Θ. Therefore, by the majoration (1), the distance between the end points of the image is inferior to

$$2\times(10\varphi(n)+1)\times(1000\delta(\Psi(5\Theta)+1))$$

in the 1-skeleton of the universal cover of P_Γ. ∎

3.4.4 Image of the leaf λ. We need a lemma from [Del].

LEMMA 3.9 ([Del] Lemma III.4): *Let L be a connected graph, L_1 be its 1-skeleton, and E a metric space. Let $h\colon L\to E$ be a continuous map. Let E' be a subset of E. Assume that:*

1. *For each edge l in L_1, $h(l)$ is homotopic in E, with fixed ends, to a curve in E' of length less than the constant M.*
2. *There exists a finite set of edges $L_1'\subset L_1$ such that any path without loop, made of consecutive edges $l_1,\dots,l_k$ in $L_1\setminus L_1'$, has its image by h homotopic in E (with fixed ends) to a curve in E' of length less than M.*

Then, for each vertex s of L, $h_*(\pi_1(L,s))$ is generated by curves in E' of length inferior to $(4\,\mathrm{Card}(L_1')+3)\times M$.

Proof: Let T be a maximal tree in L. The group $h_*(\pi_1(L,s))$ is generated by the images of the loops of the form $[s,s']e[s'',s]$, where the segments $[s,s']$ and $[s'',s]$ are in T, and where e is an edge from s' to s'' in L. In particular, the paths $[s,s']$ and $[s'',s]$ do not contain any loop, and contain at most $\mathrm{Card}(L_1')$ edges of L_1'. Each of those two segments are the concatenation of at most $\mathrm{Card}(L_1')+1$ segment without loop made of consecutive edges in $L_1\setminus L_1'$, with at most $\mathrm{Card}(L_1')$ edges of L_1'. Therefore, the image of $[s,s']$ by h is homotopic in E, with fixed ends, to a curve in E' of length less than $(2\,\mathrm{Card}(L_1')+1)\times M$, and the same is true for the image of $[s'',s]$. Finally, the image of the edge e is homotopic with fixed ends to a curve of E' of length at most M. This gives the result. ∎

Finally, we can prove Theorem 3.3. Given a morphism $h\colon G \to \Gamma$ without accidental parabolics, we set $E = P_\Gamma$, E' its 1-skeleton, and $L = \lambda$ the singular leaf of Λ given by Corollary 3.5. We choose L'_1 to be the set of singular arcs in a triangle: arcs joining two markings of a hole of a triangle, via the singular point of this triangle. Let $M = 20000\delta(\varphi(n)+1)(\Psi(5\Theta)+1)$ (which is greater than $20000\delta(\Psi(5\Theta)+1) \times n$). By Lemma 3.7 and Lemma 3.8, the assumptions of the previous lemma are fulfilled. We get that $h(G)$ is conjugated to a subgroup of Γ generated by curves in the 1-skeleton of P_Γ of length bounded by

$$(4 \times n \times (30\varphi(n))^2 + 3) \times M.$$

There are finitely many such curves. Hence, there are finitely many such subgroups, therefore this implies Theorem 3.3. ∎

4. Appendix: Coarse-piecewise-geodesics are λ-quasi-geodesics

In this appendix, we give a simple proof that coarse piecewise geodesics (Definition 2.1) are λ-quasi-geodesics (Proposition 4.2). Let $\mathcal{K}$ be a hyperbolic graph, and l a constant greater than μ (see section 2 for the constants). Let $f\colon [a,b] \to \mathcal{K}$ be a coarse piecewise geodesic, for the subdivision of $[a,b]$: $a = c_1 \leq d_1 \leq \cdots \leq c_n \leq d_n = b$.

LEMMA 4.1: *Let i be an integer in $[1,n]$. Let $t \in [c_i, d_i] \subset [a,b]$ be such that $|t - c_i| \geq 4\epsilon$ and $|t - d_i| \geq 4\epsilon$. Then $f(t)$ is at distance at most 2δ from a geodesic segment $[f(a), f(b)]$.*

The proof is similar to that of Lemma 2.17.

Proof: As $f|_{[c_i,d_i]}$ is a μ-local geodesic, the restriction $f|_{[(t-4\epsilon),(t+4\epsilon)]}$ is a geodesic segment whose ends are at distance at most 2ϵ from a segment $[f(a), f(b)]$. Let w_1 and w_2 be points in this segment realizing the minimal distance to $f(t-4\epsilon)$, $f(t+4\epsilon)$. The triangle $(f(t-4\epsilon), w_1, f(t+4\epsilon))$ and $(w_1, w_2, f(t+4\epsilon))$ are δ-thin, therefore v is at distance at most 2δ from $[w_1, w_2]$. ∎

PROPOSITION 4.2: *Let t_1 and t_2 be such that $a \leq t_1 < t_2 \leq b$. Then $|f(t_1) - f(t_2)| \geq \frac{1}{\lambda}|t_1 - t_2|$.*

Proof: Either there is a number u_1 such that $|u_1 - t_1| \leq 5\epsilon$ and such that u_1 satisfies the hypothesis of Lemma 4.1, or $|a - t_1| \leq 5\epsilon$ (in this case we write

$u_1 = a$). In both cases, $f(u_1)$ is at distance at most 2δ from a point v_1 in $[f(a), f(b)]$. Let k be a positive integer such that $t_1 + 1000k\lambda\delta \leq t_2$. Then, there exists u_{k+1}, a number such that $|t_1 + 1000k\lambda\delta - u_{k+1}| \leq 5\epsilon$, and satisfying the hypothesis of Lemma 4.1. Therefore, there exists v_{k+1} on $[f(a), f(b)]$ at distance at most 2δ from $f(u_{k+1})$.

Let m be the maximal number such that $t_1 + 1000m\lambda\delta \leq t_2$.

By definition of coarse-piecewise-geodesics, for all $k \in [1, m+1]$, $f|_{[u_k, u_{k+1}]}$ is a $\lambda/2$-quasi-geodesic. Therefore, $|f(u_k) - f(u_{k+1})| \geq \frac{2}{\lambda}|u_{k+1} - u_k|$. We deduce that $|v_k - v_{k+1}| \geq \frac{2}{\lambda}|u_{k+1} - u_k| - 4\delta$. Therefore, by summing, $|v_1 - v_{m+1}| \geq \frac{2}{\lambda} \times |u_{m+1} - u_1| - 4m\delta$.

Moreover, $|v_{m+1} - f(t_2)| \leq 5\epsilon + 1000\lambda\delta + 2\delta$, and $|v_1 - f(t_1)| \leq 5\epsilon + 2\delta$. Therefore, $|f(t_1) - f(t_2)| \geq \frac{2}{\lambda} \times |u_{m+1} - u_1| - 4(m+1)\delta - 10\epsilon - 1000\lambda\delta$. Since $|u_0 - t_1| \leq 5\epsilon$ and $|u_{m+1} - t_2| \leq 5\epsilon + 1000\lambda\delta$, we get that $|f(t_1) - f(t_2)| \geq \frac{2}{\lambda}\times(|t_2 - t_1| - 10\epsilon - 1000\lambda\delta) - 4(m+1)\delta - 10\epsilon - 1000\lambda\delta$. Since $m \leq |t_2 - t_1|/1000\lambda\delta$ and since $\lambda/2 \geq 1$, we deduce that

$$|f(t_1) - f(t_2)| \geq \left(\frac{2}{\lambda} - \frac{4\delta}{1000\lambda\delta}\right) \times |t_2 - t_1| - 20 \times (\epsilon + 100\lambda\delta).$$

Finally, one has $|f(t_1) - f(t_2)| \geq \frac{1}{\lambda}|t_2 - t_1| \times (2 - \frac{1}{250}) + (20\epsilon + 2000\lambda\delta)$. If $|t_2 - t_1| \geq 40\lambda \times (\epsilon + 100\lambda\delta)$, then $|f(t_1) - f(t_2)| \geq \frac{1}{\lambda}|t_2 - t_1|$. Otherwise, the result comes from the assumption that f is a local quasi-geodesic. ∎

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